Mohammad Ashraf, Vincenzo De Filippis, Syed Tariq Rizvi (Eds.)

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Algebra and Its Applications

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Preface

In modern world, mathematics, particularly algebra, has come to play an important role. From computer science to coding theory and cryptography, among many areas, all utilize algebra as a useful tool. While algebra is a vast subject, it is our belief that there are important connections between various branches of algebra. To emphasize those connections, the Department of Mathematics at Aligarh Muslim University, Aligarh, India, held an International Conference on Algebra and its Applications (ICAA-16) during November 12–14, 2016, under the UGC-DRS (SAP-II) programme. The conference provided an important forum to bring together experts from India and abroad in various fields of algebra and enabled younger mathematicians to benefit from their interactions with the experts. The conference included 12 plenary talks and 26 invited talks from world renowned algebraists who not only provided new research results in their talks but also presented open problems for future investigations and research. There were 72 research paper presentations from enthusiastic younger mathematicians. The well-attended conference had over two dozen participants from 18 countries including India and it served most of its objectives. The countries represented included Brazil, China, Egypt, France, Germany, Guatemala, Iran, Italy, Japan, South Korea, Morocco, Saudi Arabia, Slovenia, Spain, Taiwan, Turkey, United Arab Emirates and the U. S. A. We are thankful to all participants for their active participation. It is hoped that the conference will help foster future new connections and possible national and international collaborations. This refereed volume consists of research papers by renowned algebraists and invited speakers as well as some others who could not make it to the conference. All research papers were peer refereed by experts in various subjects. The papers present the latest research work being done on the frontiers of the various branches of algebra as well as showcase the cross-fertilization of the ideas and connections between these branches. The various topics covered in this volume include derivations in rings, category theory, Baer module theory, coding theory, graph theory, semi-group theory, HNP rings,Leavitt path algebras, generalized matrix algebras, Nakayama conjecture, near ring theory and lattice theory. The variety of the topics, methodologies and the depth of the research presented distinguishes this research volume from many others. We believe the volume will be useful not only to experts but also to the beginners of research in algebra.

We are very grateful to the following institutions and agencies for the partial support they provided to this conference: Aligarh Muslim University (AMU), Aligarh, Department of Science and Technology (DST), New Delhi, Indian National Science Academy(INSA), New Delhi, and the National Board of Higher Mathematics (NBHM), Bombay.

We are indebted to the expert referees who meticulously, and promptly, provided their reports to us despite their busy schedules.

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Without the help of our Mathematics Department colleagues, an international conference of this magnitude could not have been successful. We express our gratitude to the chairman of the department, Professor Mursaleen and all other faculty members of the department who actively participated to make the conference the success it was.

It will be injustice, if we do not express our many thanks to the research scholars of the department and other colleagues, especially in algebra, who worked hard to help take care of the many aspects of the conference. Professor M. A. Quadri's help and guidance, in spite of his health problems, was an inspiration to us and for which we express our gratitude to him.While we will like to list a number of names, we are especially grateful to Prof. Asma Ali, Dr. Nadeem ur Rehman, Dr. Shakir Ali, Dr. Mujeebur Rehamn, Dr. M. Aslam Siddeeque, Dr. Ghulam Mohammad and Dr. Aisha Jabeen, whose hard work before, during and after the conference, helped make it so successful.

This volume would not have been possible without the active cooperation we received from De Gruyter. We express our thanks to Dr. Apostolos Damialis, Ms. Nadja Schedensack and Ms. Nancy Christ for their helpful cooperation which made the publication of this volume easier.

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Contents

Asma Ali and Md. Hamidur Rahaman

On Structure of $*$-Prime Rings with Generalized Derivation

Abstract: In this article we investigate structure of a $*$–prime ring R satisfying certain algebraic identities with generalized derivation $F: R \rightarrow R$. Moreover we characterize F.

Keywords: Generalized derivations; Involutions; Prime rings; $*$-Prime rings.

1 Introduction

In all that follows, unless stated otherwise, R will be an associative ring with centre $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the Lie commutator $xy - yx$ and Jordan commutator $xy + yx$, respectively. An additive map $*: R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$. A ring equipped with an involution is called a $*$-ring or a ring with involution. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies either $a = 0$ or $b = 0$ and $*$-prime if for any $a, b \in R$, $a^*Rb = aRb = \{0\}$ or $aRb^* = aRb = \{0\}$ implies that either $a = 0$ or $b = 0$. An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. Collection of skew-hermitian elements and hermitian elements of R will be denoted by $S(R)$ and $H(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq \{0\}$. By a derivation we mean an additive mapping $d: R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ for all $x,\ y \in R$. An additive map $F: R \longrightarrow R$ is a generalized derivation if their exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x,\ y \in R$. If $d = 0$, then we have $F(xy) = F(x)y$ for all $x, y \in R$ which is called left multiplier mapping of R. Thus generalized derivations generalize both the concepts, derivation as well as left multiplier mapping. A well-known theorem of Posner [8] states that if R is prime ring and d is a derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. This result of Posner was generalized in many directions by several authors in [1, 2, 3, 6, 7] and references therein. They studied the relationship between the structure of prime or semiprime ring and the behaviour of additive maps satisfying various conditions.

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In [4] Daif and Bell proved that if R is a semiprime ring with a nonzero ideal I and d is a derivation of R such that $d\Big([x, y]\Big) = \pm[x, y]$ for all $x, y \in I$, then I is central. In particular if $I = R$, then R is commutative. Later in [9], Quadri et al. discussed the commutativity of prime rings with generalized derivations. More precisely, they proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F\Big([x, y]\Big) = [x, y]$ for all $x, y \in I$, then R is commutative. Further, this result of Quadri et al. is studied in semiprime ring by Dhara in [5]. He proved that if R is a semiprime ring with a nonzero ideal I and F is a generalized derivation of R associated with a derivation d of R such that $d(I) \neq 0$ and $F\Big([x, y]\Big) = \pm[x, y]$ or $F(x \circ y) = \pm(x \circ y)$ for all $x, y \in I$, then R contains a nonzero central ideal. In case R is a prime ring then, R must be commutative, provided $d \neq 0$.

In the present paper, we study the situations when (i) $F(x \circ x^*) = a(x \circ x^*)$, (ii) $F([x, x^*]) = a[x, x^*]$, (iii) $[F(x), F(x^*)] = a[x, x^*]$, (iv) $F(x) \circ F(x^*) = a(x \circ x^*)$ for all $x, y \in I$, a nonzero left ideal of a $*$–prime ring R with generalized derivation F and $a \in \{0, \pm1\}$.

2 Preliminary results

Before starting our results, we state some well known facts which are very crucial for developing the proof of our main results.

Fact 2.1. *If R is a $2-torsion$ free $*$–prime ring and $H(R) \cap Z(R) \neq \{0\}$, then $d(x) = 0$ for all $x \in H(R) \cap Z(R)$ implies that $d(x) = 0$ for all $x \in Z(R)$. Indeed if $d(x) = 0$ for all $x \in H(R) \cap Z(R)$, replacing x by y^2 where $y \in S(R) \cap Z(R)$, then we get $d(y)y = 0$ for all $y \in S(R) \cap Z(R)$, so $d(y) = 0$ for all $y \in S(R) \cap Z(R)$. As conclusion, we have $d(y) = 0$ for all $y \in Z(R)$.*

Fact 2.2. *If R be a $*-prime$ ring with involution of second kind, then $Z(R) \cap H(R) \neq \{0\}$.*

Also throughout the present paper we shall make use of the following identities without any specific mention: For all $x, y, z \in R$;

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z$$

$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$$

$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z]\ .$$

3 Main Results

Theorem 3.1. Let R be a $2-torsion$ free $*$-prime ring with involution of second kind and I a nonzero left ideal of R such that $I^* \subseteq I$. If R admits a generalized derivation

F with associated derivation d such that $F(x \circ x^*) = a(x \circ x^*)$ for all $x \in I$, where $a \in \{0, \pm 1\}$, then one of the following holds:
(i) $Id(I) = \{0\}$;
(ii) R is commutative and $F(r) = ar$ for all $r \in R$.

Proof. By hypothesis

$$F(x \circ x^*) = a(x \circ x^*) \text{ for all } x \in I\,. \tag{1}$$

Replacing x by $x + y$, we have

$$F(x \circ y^*) + F(y \circ x^*) = a(x \circ y^*) + a(y \circ x^*) \text{ for all } x, y \in I\,. \tag{2}$$

Substituting y^* for y, we get

$$F(x \circ y) + F(y^* \circ x^*) = a(x \circ y) + a(y^* \circ x^*) \text{ for all } x, y \in I\,. \tag{3}$$

Replacing y by yh where $h \in Z(R) \cap H(R) \setminus \{0\}$ and using (3), we get

$$(x \circ y + y^* \circ x^*)d(h) = 0 \text{ for all } x, y \in I\,. \tag{4}$$

This can be written as $(x \circ y + y^* \circ x^*)Rd(h) = \{0\} = (x \circ y + y^* \circ x^*)^* Rd(h)$. Since R is a $* - prime$ ring, we have either $x \circ y + y^* \circ x^* = 0$ or $d(h) = 0$.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, by Fact 2.1, we get $d(z) = 0$ for all $z \in Z(R)$. Replacing y by ys in (3), $s \in Z(R) \cap S(R) \setminus \{0\}$, we get

$$(F(x \circ y) - F(y^* \circ x^*) - a(x \circ y) + a(y^* \circ x^*))s = 0 \text{ for all } x, y \in I\,. \tag{5}$$

That is

$$F(x \circ y) - F(y^* \circ x^*) = a(x \circ y) - a(y^* \circ x^*) \text{ for all } x, y \in I\,. \tag{6}$$

Equation (3) and (6) together give that

$$F(x \circ y) = a(x \circ y) \text{ for all } x, y \in I\,. \tag{7}$$

Replacing y by yx in the above equation (7), we have

$$F((xoy)x) = a((x \circ y)x) \text{ for all } x, y \in I\,. \tag{8}$$

That is

$$F(xoy)x + (xoy)d(x) = a((x \circ y)x) \text{ for all } x, y \in I\,.$$

Using equation (8), we get

$$(xoy)d(x) = 0 \text{ for all } x, y \in I\,. \tag{9}$$

Again we replace y by zy, $z \in I$ in equation (9), to have

$$[x, z]yd(x) = 0 \text{ for all } x, y, z \in I\,. \tag{10}$$

Since I is a left ideal, it follows that $[x, z]Ryd(x) = \{0\} = [x, z]^* Ryd(x)$ for all $x, y \in I$. Since R is a $*$-prime ring, either $[x, z] = 0$ or $yd(x) = 0$. Now $yd(x) = 0$ that is $Id(I) = \{0\}$ which is part (i). Let $[x, z] = 0$ for all $x, z \in I$. Replacing z by $r_1 z$, where $r_1 \in R$ we have $[x, r_1]z = 0$ i.e $[x, r_1]Rz = \{0\} = [x, r_1]Rz^*$. Since R is a $* - prime$ ring, we have either $[x, r_1] = 0$ or $I^* = \{0\}$. If $I^* = \{0\}$, then $I = \{0\}$, a contradiction. So we have $[x, r_1] = 0$. Again replacing x by $r_2 x$, where $r_2 \in R$ we have $[r_2, r_1]x = 0$ that is $[r_2, r_1]Rx = \{0\} = [r_2, r_1]Rx^* = 0$. Since R is a $* - prime$ ring, we have $[r_2, r_1] = 0$ that is R ia commutative. Then equation (7) becomes $F(xy) = axy$ for all $x, y \in I$. This gives that

$$F(x)y + xd(y) - axy = (F(x) - ax)y + xd(y) = 0 \text{ for all } x, y \in I\,. \tag{11}$$

Since R is commutative, $xr \in I$. Replacing x by xr in equation (11), we get $0 = (F(x) - ax)ry + xrd(y) + xd(r)y = \{(F(x) - ax)y + xd(y))\}r + d(r)yx$. Using (11), we find that $d(R)I^2 = \{0\}$, implying that $d(R) = \{0\}$. Then from equation (11), we have $(F(x)-ax)I = \{0\}$, which gives that $F(x) = ax$ for all $x \in I$. Replacing x by rx, $r \in R$ and using $d(R) = \{0\}$, we get $F(r) = ar$ for all $r \in R$. If $x \circ y + y^* \circ x^* = 0$ for any $x, y \in I$, then replacing y by yk, where $k \in Z(R) \cap S(R) \setminus \{0\}$, we find that $(x \circ y - y^* \circ x^*)k = 0$ that is $x \circ y - y^* \circ x^* = 0$. Together these two equation give that $x \circ y = 0$ for all $x, y \in I$. Replacing y by $r_1 y$, where $r_1 \in R$ we have $[x, r_1]y = 0$ i.e $[x, r_1]Ry = \{0\} = [x, r_1]Ry^*$. Since R is a $* - prime$ ring, we have either $[x, r_1] = 0$ or $I^* = \{0\}$. If $I^* = \{0\}$, then $I = \{0\}$, a contradiction. So we have $[x, r_1] = 0$. Again replacing x by $r_2 x$, where $r_2 \in R$ we have $[r_2, r_1]x = 0$ that is $[r_2, r_1]Rx = \{0\} = [r_2, r_1]Rx^* = 0$. Since R is a $* - prime$ ring, we have $[r_2, r_1] = 0$ i.e R is commutative. □

The following examples demonstrate that R to be $* - prime$ can not be omitted in the hypothesis of Theorem 3.1

Example 1. *Let* $R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in Z \right\}$ *and* $I = R$. *Define maps* $F, d, * : R \to R$ *by* $F(\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & -z \\ 0 & 0 & 0 \end{pmatrix}$, $d(\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & z & -y \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix}$.

It can be verified that $Z(R) \cap S(R) \neq \{0\}$ *and* $F(x \circ x^*) = a(x \circ x^*)$ *for all* $x \in I$. *However, neither* $Id(I) = \{0\}$ *nor* R *is commutative.*

Theorem 3.2. Let R be a $2-torsion$ free ∗-prime ring with involution of second kind and I a nonzero left ideal of R such that $I^* \subseteq I$. If R admits a generalized derivation F with associated derivation d which commutes with ∗ such that $F(x) \circ d(x^*) = a(x \circ x^*)$ for all $x \in I$, where $a \in \{0, \pm 1\}$, then either $F(I) \subseteq Z(R)$ *or* $[[F(x), x], d(x)] = 0$. *In particular if I is an ideal, then either R is commutative or F acts as a left multiplier.*

Proof. By hypothesis

$$F(x) \circ d(x^*) = a(x \circ x^*) \text{ for all } x \in I\,. \tag{12}$$

Linearization of the above equation (12) yields that

$$F(x) \circ d(x^*) + F(x) \circ d(y^*) + F(y) \circ d(x^*) + F(y) \circ d(y^*)$$

$$= a(x \circ x^*) + a(x \circ y^*) + a(y \circ x^*) + a(y \circ y^*) \text{ for all } x, y \in I\,. \tag{13}$$

Using equation (12), we have

$$F(x) \circ d(y^*) + F(y) \circ d(x^*) = a(x \circ y^*) + a(y \circ x^*) \text{ for all } x, y \in I\,. \tag{14}$$

Writing y^* in the place of y, we have

$$F(x) \circ d(y) + F(y^*) \circ d(x^*) = a(x \circ y) + a(y^* \circ x^*) \text{ for all } x, y \in I\,. \tag{15}$$

Replacing y by ys, $s \in Z(R) \cap H(R) \setminus \{o\}$ and using equation (15), we find that

$$(F(x) \circ y + y^* \circ d(x^*))d(s) = 0 \text{ for all } x, y \in I\,. \tag{16}$$

This can be written as $(F(x)\circ y+y^*\circ d(x^*))Rd(s) = 0 = (F(x)\circ y+y^*\circ d(x^*))Rd(s)^*$. Since R is a $*-prime$ ring, we have either $F(x)\circ y+y^*\circ d(x^*) = 0$ for all $x, y \in I$ or $d(Z(R)) = \{0\}$. If $F(x) \circ y + y^* \circ d(x^*) = 0$ for all $x, y \in I$, replacing y by yk, $k \in Z(R) \cap S(R) \setminus \{0\}$, we have $F(x) \circ y - y^* \circ d(x^*) = 0$ for all $x, y \in I$. Adding these two equations, we have

$$F(x)y + yd(x) = 0 \text{ for all } x, y \in I\,. \tag{17}$$

Replacing y by ry, $r \in R$ and using $yF(x) = -F(x)y$, we get

$$[F(x), r]y = 0 \text{ for all } x, y \in I \textit{ and } r \in R\,. \tag{18}$$

That is

$$[F(x), R]RI = \{0\} = [F(x), R]RI^*\,. \tag{19}$$

Since R is a $* - prime$ ring and I is nonzero left ideal, we have

$$[F(x), r] = 0 \text{ for all } x \in I \textit{ and } r \in R\,. \tag{20}$$

That is $F(I) \subseteq Z(R)$. In particular if I is an ideal, then replacing x by xr in equation (20), we find that $[xd(r), r] = 0$. Again replacing x by zx, $z \in R$, we find that $[z, r]RId(r) = \{0\} = [z, r]^* RId(r)$. Since R is a $*$-prime ring and I is nonzero ideal, we have either R is commutative or F acts as a left multiplier. If $d(k) = 0$ for all $k \in Z(R) \cap H(R)$, by using Fact 2.1, we get $d(z) = 0$ for all $z \in Z(R)$. Replacing y by ys, where $s \in Z(R) \cap S(R) \setminus \{0\}$ in (15), we get

$$F(x) \circ d(y) - F(y^*) \circ d(x^*) = a(x \circ y) - a(y^* \circ x^*) \text{ for all } x, y \in I\,. \tag{21}$$

Now adding equation (15) and (21), we get

$$F(x) \circ d(y) = a(x \circ y) \text{ for all } x, y \in I\,. \tag{22}$$

Replacing y by yx in the above expression, we have for all $x, y \in I$

$$F(x)(d(y)x + yd(x)) + (d(y)x + yd(x))F(x)) = a((x \circ y)x)\,. \tag{23}$$

Now using equation (22) in equation (23), we get

$$F(x)yd(x) + d(y)[x, F(x)] + yd(x)F(x) = 0\,. \tag{24}$$

That is

$$d(y)[F(x), x] + [F(x), y]d(x) = 0\,. \tag{25}$$

Again replacing y by xy in the above equation (25) and using (25) we have

$$d(x)y[F(x), x] + [F(x), x]yd(x) = 0 \tag{26}$$

Replacing y by $d(x)y$ in the above expression, we have

$$d(x)^2y[F(x), x] + [F(x), x]d(x)yd(x) = 0 \tag{27}$$

left multiplying equation (26) by $d(x)$ and subtracting from equation (27), we have

$$[[F(x), x], d(x)]yd(x) = 0 \text{ for all } x, y \in I\,. \tag{28}$$

This can be written as $[[F(x), x], d(x)]RId(x)I = \{0\} = [[F(x), x], d(x)]R(Id(x)I)^*$. Since R is a $* - prime$ ring and I is nonzero, we have $[[F(x), x], d(x)] = 0$. □

Theorem 3.3. Let R be a $2 - torsion$ free $*$-prime ring with involution of second kind and I a nonzero left ideal of R such that $I^* \subseteq I$. If R admits a generalized derivation F with associated derivation d such that $F([x, x^*]) = a[x, x^*]$ for all $x \in I$, where $a \in \{0, \pm 1\}$, then one of the following holds:
(i) $I[I, I] = \{0\}$;
(ii) R is commutative;
(iii) $F(x) = ax$ for all $x \in I$.

Proof. By hypothesis

$$F([x, x^*]) = a[x, x^*] \text{ for all } x \in I\,. \tag{29}$$

Replacing x by $x + y$, we have

$$F([x, y^*]) + F([y, x^*]) = a[x, y^*] + a[y, x^*] \text{ for all } x, y \in I\,. \tag{30}$$

Writting y^* in place of y, we get

$$F([x, y]) + F([y^*, x^*]) = a[x, y] + a[y^*, x^*] \text{ for all } x, y \in I\,. \tag{31}$$

Replacing y by yk where $k \in Z(R) \cap H(R) \setminus \{0\}$ and using (31), we get

$$([x, y] + [y^*, x^*])d(k) = 0 \text{ for all } x, y \in I\,. \tag{32}$$

This can be written as $([x, y] + [y^*, x^*])Rd(k) = \{0\} = ([x, y] + [y^*, x^*])^* Rd(k)$. Since R is a $* - prime$ ring, we have either $[x, y] + [y^*, x^*] = 0$ or $d(k) = 0$.

If $[x, y] + [y^*, x^*] = 0$ for any $x, y \in I$, then replacing y by yk, where $k \in Z(R) \cap S(R) \setminus \{0\}$, we find that $([x, y] - [y^*, x^*])k = 0$ that is $[x, y] - [y^*, x^*] = 0$. Together these two equation gives that $[x, y] = 0$ for all $x, y \in I$. Replacing y by $r_1 y$, where $r_1 \in R$ we have $[x, r_1]y = 0$ i.e $[x, r_1]Ry = \{0\} = [x, r_1]Ry^*$. Since R is a $* - prime$ ring, we have either $[x, r_1] = 0$ or $I^* = \{0\}$. If $I^* = \{0\}$, then $I = \{0\}$, a contradiction. So we have $[x, r_1] = 0$. Again replacing x by $r_2 x$, where $r_2 \in R$ we have $[r_2, r_1]x = 0$ that is $[r_2, r_1]Rx = \{0\} = [r_2, r_1]Rx^* = 0$. Since R is a $* - prime$ ring, we have $[r_2, r_1] = 0$ i.e R is commutative.

If $d(k) = 0$ for all $k \in Z(R) \cap H(R)$, by using Fact 2.1, we get $d(z) = 0$ for all $z \in Z(R)$. Replacing y by ys in (31), $s \in Z(R) \cap S(R) \setminus \{0\}$, we get

$$(F([x, y]) - F([y^*, x^*]) - a[x, y] + a[y^*, x^*])s = 0 \text{ for all } x, y \in I\,. \tag{33}$$

That is

$$F([x, y]) - F([y^*, x^*] = a[x, y] - a[y^*, x^*] \text{ for all } x, y \in I\,. \tag{34}$$

Equation (31) and (34) together gives that

$$F([x, y]) = a[x, y] \text{ for all } x, y \in I\,. \tag{35}$$

Replacing x by yx, we get

$$F([x, y])y + [x, y]d(y) = a[x, y]y \text{ for all } x, y \in I\,. \tag{36}$$

Now using (35), the above equation yields that $[x, y]d(y) = 0$. Again replacing x by zx, we get $[z, y]xd(y) = 0$ that is $[z, y]RId(y) = \{0\}$ for all $x, y, z \in I$. Since R is a $*$-prime ring, we find that either $[z, y] = 0$ for all $z, y \in I$ or $Id(I) = \{0\}$ that is R is commutative or $Id(I) = \{0\}$. Now let R be noncommutative. Then for any $x, y \in I$,

$F(xy) = F(x)y + xd(y) = F(x)y$ that is F acts as left multiplier on I. Then for any $x, y, z \in I$, replacing y by yz in equation (35), we get

$$F([x, y]z + y[x, z]) = a\{[x, y]z + y[x, z]\} \tag{37}$$

Since F acts as a left multiplier on I, we have

$$F([x, y])z + F(y)[x, z] = a\{[x, y]z + y[x, z]\} \tag{38}$$

Now using equation (35), we get $(F(y) - ay)[x, z] = 0$. Again replacing y by yu, $u \in I$, we find that $(F(y) - ay)u[x, z] = 0$ which gives that $(F(y) - ay)RI[x, z] = \{0\}$. Since R is a $* - prime$ ring, either $F(y) = ay$ for all $y \in I$ or $I[I, I] = \{0\}$. □

Theorem 3.4. Let R be a $2 - torsion$ free $*$-prime ring with involution of second kind and I a nonzero left ideal of R such that $I^* \subseteq I$. If R admits a generalized derivation F with associated derivation d which commutes with $*$ such that $[F(x), d(x^*)] = a[x, x^*]$ for all $x \in I$, where $a \in \{0, \pm 1\}$, then either $F(I) \subseteq Z(R)$ or $[[F(x), x], d(x)] = 0$. In particular if I is an ideal, then either R is commutative or F acts as a left multiplier.

Proof. By hypothesis

$$[F(x), d(x^*)] = a[x, x^*] \text{ for all } x \in I\,. \tag{39}$$

Linearization of the above equation (39) yields that

$$[F(x), d(x^*)] + [F(x), d(y^*)] + [F(y), d(x^*)] + [F(y), d(y^*)]$$

$$= a[x, x^*] + a[x, y^*] + a[y, x^*] + a[y, y^*] \text{ for all } x, y \in I\,. \tag{40}$$

Using equation (39) in the above equation (40), we have

$$[F(x), d(y^*)] + [F(y), d(x)^*)] = a[x, y^*] + a[y, x^*] \text{ for all } x, y \in I\,. \tag{41}$$

Writing y^* in the place of y, we have

$$[F(x), d(y)] + [F(y^*), d(x^*)] = a[x, y] + a[y^*, x^*] \text{ for all } x, y \in I\,. \tag{42}$$

Replacing y by ys, $s \in Z(R) \cap H(R)$ $\{0\}$ and using equation (42), we find that

$$([F(x), y] + [y^*, d(x^*)])d(s) = 0 \text{ for all } x, y \in I\,. \tag{43}$$

This can be written as

$$([F(x), y] + [y^*, d(x^*)])Rd(s) = \{0\} = ([F(x), y] + [y^*, d(x^*)])Rd(s)^*\,.$$

Since R is a $* - prime$ ring, we have either $[F(x), y] + [y^*, d(x^*)] = 0$ for all $x, y \in I$ or $d(s) = 0$ for all $s \in Z(R) \cap H(R)$. If $[F(x), y] + [y^*, d(x^*)] = 0$, then replacing y by yk,

$k \in Z(R) \cap S(R) \setminus \{0\}$, we have $[F(x), y] - [y^*, d(x^*)] = 0$ for all $x, y \in I$. Adding them, we have

$$F(x)y - yF(x) = 0 \text{ for all } x, y \in I\,. \tag{44}$$

Replacing y by ry, $r \in R$ and using $yF(x) - F(x)y = 0$, we get

$$[F(x), r]y = 0 \text{ for all } x, y \in I \textit{ and } r \in R\,. \tag{45}$$

Replacing y by wy, $w \in R$, we find that

$$[F(x), r]RI = \{0\} = [F(x), r]RI^* \text{ for all } x \in I \textit{ and } r \in R\,. \tag{46}$$

Since R is a $*-prime$ ring and I is nonzero, we have $[F(x), R] = \{0\}$ that is $F(I) \subseteq Z(R)$. In particular if I is an ideal, then replacing x by xr in equation (46), we find that $[xd(r), r] = 0$. Again replacing x by zx, $z \in R$, we obtain that $[z, r]RId(r) = \{0\} = [z, r]^*RId(r)$. Since R is $*$-prime ring and I is nonzero ideal, we have either R is commutative or F acts as a left multiplier. If $d(k) = 0$ for all $k \in Z(R) \cap H(R)$, by using Fact 2.1, we get $d(z) = 0$ for all $z \in Z(R)$. Replacing y by ys, where $s \in Z(R) \cap S(R) \setminus \{0\}$ in (42), we get

$$[F(x), d(y)] - [F(y^*), d(x^*)] = a[x, y] - a[y^*, x^*] \text{ for all } x, y \in I\,. \tag{47}$$

Now adding equation (42) and (47), we get

$$[F(x), d(y)] = a[x, y] \text{ for all } x, y \in I\,. \tag{48}$$

Replacing y by yx in the above expression, we have for all $x, y \in I$

$$[F(x), d(y)]x + d(y)[F(x), x] + [F(x), y]d(x) + y[F(x), d(x)] = a[x, y]x\,. \tag{49}$$

Now using equation (48) in equation (49), we get

$$d(y)[F(x), x] + [F(x), y]d(x) = 0\,. \tag{50}$$

Again replacing y by xy in the above equation (50) and using (50) we have

$$d(x)y[F(x), x] + [F(x), x]yd(x) = 0 \tag{51}$$

Replacing y by $d(x)y$ in the above expression, we have

$$d(x)^2y[F(x), x] + [F(x), x]d(x)yd(x = 0 \tag{52}$$

left multiplying equation (51) by $d(x)$ and subtracting from equation (52), we have

$$[[F(x), x], d(x)]yd(x) = 0 \text{ for all } x, y \in I\,. \tag{53}$$

This can be written as $[[F(x), x], d(x)]RId(x)I = \{0\} = [[F(x), x], d(x)]R(Id(x)I)^*$. Since R is a $* - prime$ ring and I is nonzero, we have $[[F(x), x], d(x)] = 0$. □

The following examples demonstrate that $*$ to be involution of second kind can not be omitted in the hypothesis of Theorem 3.4.

Example 2. *Let* $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z \right\}$ *and* $I = R$. *Define maps* $F, d, * : R \to R$ *by* $F(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}$, $d(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ *and* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. *Obviously* $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in Z \right\}$. *Then* $x^* = x$ *for all* $x \in Z(R)$ *and hence* $Z(R) \subseteq H(R)$, *which shows that the involution* $*$ *is of the first kind. It can be verified that* $[F(x), d(x^*)] = a[x, x^*]$ *for all* $x \in I$. *However neither* R *is commutative nor* F *acts as left multiplier. Hence, in Theorem 3.4, the hypothesis of second kind involution is crucial.*

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A characterization of additive mappings in rings with involution

Abstract: The main purpose of this paper is to characterize some additive mappings satisfying certain functional equations in rings with involution. In particular, we prove that any Jordan $*$-centralizer on a 2-torsion free semiprime $*$-ring is a reverse $*$-centralizer. As an application of this result, Jordan $*$-centralizers of semiprime rings are characterized. Further, we establish that if R is a $(m+n)!$- torsion free noncommutative prime ring with involution $*$ and D, G are Jordan $*$-derivations on R such that $D(x^m)x^n \pm x^n G(x^m) = 0$ for all $x \in R$, where m, n are non-negative integers, then $D = G = 0$. This result is in the spirit of the classical result of Posner [21], which states that: Let R be a prime ring and D a derivation of R such that $xD(x) - D(x)x = 0$ for all $x \in R$. Then R is commutative or $D = 0$.

Keywords: Prime ring; semiprime ring; involution; $*$-centralizer; reverse $*$-centralizer; Jordan $*$-centralizer $*$-derivation; Jordan $*$-derivation.

1 Introduction

Throughout this paper, R will represent an associative ring with centre $Z(R)$. We denote by $Q_m r$, Q_r, Q_s and C the maximal ring of quotients, the right Martindale ring of quotients, the symmetric Martindale ring of quotients and the extended centroid of R. For the explanation of $Q_m r$, Q_r, Q_s and C, we refer the reader to [6]. For all $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anti-commutator $xy + yx$. We shall use the basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = [x, y]z + y[x, z]$ for all $x, y, z \in R$. A ring R is said to be n-torsion free, where $n > 1$ is an integer, when $nx = 0$ implies $x = 0$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^* x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution is known

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as ring with involution or $*$-ring. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case, $S(R) \cap Z(R) \neq (0)$. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. An additive mapping $T\colon R \to R$ is called a left centralizer of R if $T(xy) = T(x)y$ holds for all $x, y \in R$. For a semiprime ring R, all left centralizers are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale right ring of quotients Q_r of R. In case R has identity element, $T\colon R \to R$ is a left centralizer if and only if T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. The definition of a right centralizer should be self-explanatory. An additive mapping T is called a two-sided centralizer in case T is a left and a right centralizer. In case $T\colon R \longrightarrow R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see [6, Theorem 2.3.2]). Some more related results concerning centralizers on rings and algebras we refer the reader to ([18, 19, 20, 22, 23, 25] and [26], where further references can be found).

Let R be a ring with involution $*$. According to [4], an additive mapping $T\colon R \longrightarrow R$ is said to be a left $*$-centralizer (resp. reverse left $*$-centralizer) if $T(xy) = T(x)y^*$ (resp. $T(xy) = T(y)x^*$) holds for all $x, y \in R$. The definition of a right $*$-centralizer (resp. reverse right $*$-centralizer) should be self explanatory. An additive mapping $T\colon R \longrightarrow R$ is called a $*$-centralizer if T is both a left and a right $*$-centralizer. Note that for some fixed element $a \in R$, the mapping $x \mapsto ax^*$ is a reverse left $*$-centralizer and $x \mapsto x^*a$ is a reverse right $*$-centralizer on R. We call Jordan centralizer T on R is an additive mapping $T\colon R \longrightarrow R$ satisfying $T(x \circ y) = T(x) \circ y = x \circ T(y)$ for all $x, y \in R$. Following the above convention, we define a Jordan $*$-centralizer to be the additive mapping $T\colon R \longrightarrow R$ such that $T(x \circ y) = T(x) \circ y^* = x^* \circ T(y)$ for all $x, y \in R$, where R is ring with involution $*$. It is easy to check that every $*$-centralizer is a Jordan $*$-centralizer. However, the converse is not true in general. The following example justifies this fact:

Example 1. *Let* $R = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} \,\middle|\, a, b \in Z_2 \right\}$. *Define mappings* $T\colon R \longrightarrow R$, *and* $*\colon R \longrightarrow R$ *such that* $T\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix}$.
Then it is easy to verify that T *is a Jordan* $*$*-centralizer on* R. *However,* T *is not a* $*$*-centralizer.*

In Section 3 besides proving some results on Jordan $*$-centralizers, we establish (Theorem 3.4) set of conditions under which every Jordan $*$-centralizer on $*$-ring is a $*$-centralizer.

Following [13], an additive mapping $D\colon R \to R$ is called a $*$-derivation (resp. Jordan $*$-derivation) if $D(xy) = D(x)y^* + xD(y)$ (resp. $D(x^2) = D(x)x^* + xD(x)$) holds for all $x, y \in R$, where R is ring with involution. Let S be a nonempty subset of R. A mapping $f\colon R \to R$ is said to be centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$, and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$. Over last some decades, several authors have investigated the relationship between the structure of the ring R and certain specific types of derivations on R. The first result in this direction is due to Posner [21], who proved that if a prime ring R admits a nonzero derivation D such that $[D(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. This result was subsequently refined and extended by a number of algebraists (see [7, 8, 10], where further references can be found). Recently, Argac [5] generalized Posner's result [21, Lemma] in the setting of semiprime ring involving pair of derivations. The last section of the present paper deals with the study of $*$-derivations and Jordan $*$-derivations satisfying certain functional equations in rings with involution. In particular, we study Argac's result [5, Theorem 3.1] for pair of Jordan $*$-derivations in the setting of prime rings with involution.

2 The necessary preliminaries

We begin with the following lemmas which we needed for developing the proof of our main results.

Lemma 2.1 ([2, Lemma 2.1]). *Let R be a prime ring with involution $'*'$, of characteristic different from 2. If $S(R) \subseteq Z(R)$, then R is commutative.*

Lemma 2.2 ([2, Theorem 3.2]). *Let R be a noncommutative prime ring with involution of the second kind such that $char(R) \neq 2$. Let D be a Jordan $*$-derivation of R such that $[D(x), x] = 0$ for all $x \in R$. Then $D = 0$.*

Lemma 2.3. *Let R be a prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. If $H(R) \subseteq Z(R)$, then R is commutative.*

Proof. We have $H(R) \subseteq Z(R)$. This gives $[h, x] = 0$ for all $h \in H(R)$ and $x \in R$. If $k \in S(R)$, then $k^2 \in H(R)$ and therefore we have $0 = [k^2, x] = [k, x]k + k[k, x]$ that is,

$$[k, x]k + k[k, x] = 0 \tag{1}$$

for all $k \in S(R)$ and $x \in R$. Replacing k by $k + k_1$, $k_1 \in S(R) \cap Z(R)$ in (1), we get $[k, x]k_1 + k_1[k, x] = 0$. That is, $2k_1[k, x] = 0$ for all $x \in R$. This further implies that $k_1[k, x] = 0$ for all $x \in R$. Now since the center of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we finally arrive at $[k, x] = 0$ for all $x \in R$. That is, $S(R) \subseteq Z(R)$. Therefore in view of Lemma 2.1, we get R is commutative. ☐

Lemma 2.4. *Let R be a semiprime ring with involution $'*'$ and $d\colon R \to R$ be additive mapping such that $d(xy) = d(y)x^* + y^* d(x)$ and $a \in R$ some fixed element.*
(i) $d(x)d(y) = 0$ for all $x, y \in R$ implies $d = 0$.
(ii) $ax^* - x^* a \in Z(R)$ for all $x \in R$ implies $a \in Z(R)$.

Proof. (i) By our hypothesis, we have

$$d(x)d(y) = 0 \text{ for all } x, y \in R\,. \tag{2}$$

Replacing y by xy in (2) and using (2), we obtain $d(x)y^* d(x) = d(x)d(xy) - d(x)d(y)x^* = 0$ for all $x, y \in R$. This implies that $d(x)Rd(x) = (0)$ for all $x \in R$. Hence the semiprimeness of R yields the required result.
(ii) We set $d(x) = ax^* - x^* a$ for all $x \in R$. Then it is easy to verify that d satisfies the relation $d(xy) = d(y)x^* + y^* d(x)$ for all $x, y \in R$. Therefore by the hypothesis we have $d(x) \in Z(R)$ for all $x \in R$. This gives $d(y)x^* = x^* d(y)$ for all $x, y \in R$. Also we have $d(yz)x^* = x^* d(yz)$ for all $x, y, z \in R$. Therefore, we obtain

$$d(z)y^* x^* + z^* d(y)x^* = x^* d(z)y^* + x^* z^* d(y)$$

for all $x, y, z \in R$. This implies that

$$d(z)[y^*, x^*] = d(y)[x^*, z^*]$$

for all $x, y, z \in R$. Replacing z by a^* in the above relation and using the fact that $d(a^*) = 0$, we obtain $d(y)[a, x^*] = 0$ for all $x, y \in R$. This implies $d(y)d(x) = 0$ for all $x, y \in R$. Hence, in view of part (i) we get the required result. ☐

Lemma 2.5. *Let R be a semiprime ring with involution $'*'$ and $a \in R$ some fixed element. If $T(x) = ax^* + x^* a$ for all $x \in R$ is a Jordan $*$-centralizer, then $a \in Z(R)$.*

Proof. Assuming T to be a Jordan $*$-centralizer, then we have $T(xy + yx) = T(x)y^* + y^* T(x)$ for all $x, y \in R$. Thus our hypothesis yields that

$$a(xy + yx)^* + (xy + yx)^* a = (ax^* + x^* a)y^* + y^* (ax^* + x^* a)$$

$$ay^* x^* + ax^* y^* + y^* x^* a + x^* y^* a = ax^* y^* + x^* ay^* + y^* ax^* + y^* x^* a$$

$$(ay^* - y^* a)x^* - x^* (ay^* - y^* a) = ax^* y^* + y^* x^* a - ax^* y^* + y^* x^* a = 0$$

for all $x, y \in R$. In view of Lemma 2.4(ii), we conclude that $a \in Z(R)$, as desired. This proves the lemma. ☐

Lemma 2.6. *Let R be a semiprime ring with involution $'*'$. Then, every Jordan $*$-centralizer of R maps $Z(R)$ into $Z(R)$.*

Proof. For any element $c \in Z(R)$, set $a = T(c)$. Then we have

$$2T(cx) = T(cx + xc) = T(c)x^* + x^* T(c) = ax^* + x^* a\,.$$

Let $S(x) = 2T(cx)$ for all $x \in R$. Then we see that

$$\begin{aligned} S(xy + yx) &= 2T(c(xy + yx)) \\ &= 2T(cxy + ycx) \\ &= 2T(cx)y^* + 2y^* T(cx) \\ &= S(x)y^* + y^* S(x) \end{aligned}$$

for all $x, y \in R$. This shows that S is also a Jordan $*$-centralizer. Hence by Lemma 2.5, we obtain $T(c) \in Z(R)$. □

Proposition 2.1. *Let R be a semiprime ring with involution $'*'$ and let $f : R \to R$ be an additive mapping. Suppose that either*

$$f(x)x^* = 0 \text{ or } x^* f(x) = 0$$

holds for all $x \in R$. In both the cases $f = 0$.

Proof. We first consider the case $f(x)x^* = 0$, for all $x \in R$. Linearizing it we have

$$f(x)y^* + f(y)x^* = 0 \tag{3}$$

for all $x, y \in R$. Replacing y by y^2 in (3), we have

$$f(x)(y^*)^2 + f(y^2)x^* = 0 \tag{4}$$

for all $x, y \in R$. Right multiplying (3) by y^*, we obtain

$$f(x)(y^*)^2 + f(y)x^* y^* = 0 \tag{5}$$

for all $x, y \in R$. Combining (4) and (5), we get

$$f(y^2)x^* - f(y)x^* y^* = 0 \tag{6}$$

for all $x, y \in R$. Substituting $f(y)^* x$ for x in (6), we find that

$$f(y^2)x^* f(y) - f(y)x^* f(y)y^* = 0$$

for all $x, y \in R$. In view of our hypothesis, we conclude that

$$f(y^2)x^* f(y) = 0 \tag{7}$$

for all $x, y \in R$. Right multiplying (6) by $f(y)$ gives because of (7)

$$f(y)x^* y^* f(y) = 0$$

for all $x, y \in R$. This further implies that $y^* f(y)x^* y^* f(y) = 0$ for all $x, y \in R$. The semiprimeness of R yields that

$$x^* f(x) = 0 \tag{8}$$

for all $x \in R$. Right multiplying (3) by $f(x)$ gives because of (8) that $f(x)y^* f(x) = 0$ for all $x, y \in R$. Thus by the semiprimeness of R, we conclude that $f(x) = 0$ for all $x \in R$. □

By the same argument, we obtain the similar conclusion in the case $x^*f(x) = 0$ for all $x \in R$. This proves the lemma completely.

Proposition 2.2. *Let R be a 2-torsion free semiprime ring with involution $'*'$. Suppose there exists an additive mapping $T\colon R \to R$ such that $T(xyx) = x^*T(y)x^*$ for all $x, y \in R$. Then T is a reverse $*$-centralizer.*

Proof. Our hypothesis yields that $T(xyx)^* = xT(y)^*x$ for all $x, y \in R$. Taking $S(x) = T(x)^*$, then we get $S(xyx) = xS(y)x$ for all $x, y \in R$. In view of [23, Theorem 1], we conclude that S is a centralizer on R. That is, $S(xy) = S(x)y = xS(y)$ for all $x, y \in R$. This intern implies $T(xy) = y^*T(x) = T(y)x^*$ for all $x, y \in R$. Thereby showing T is a reverse $*$-centralizer of R. □

Proposition 2.3. *Let R be a semiprime ring with involution $'*'$ and let $f, g\colon R \to Q_{mr}$ be additive mappings. If $y^*f(x) + g(y)x^* = 0$ for all $x, y \in R$, then there exists a unique $q \in Q_{mr}$ such that $f(x) = -px^*$ and $g(x) = x^*p$ for all $x \in R$, where $p = q^*$.*

Proof. By the assumption we have $y^*f(x) + g(y)x^* = 0$ for all $x, y \in R$. This implies $f(x)^*y + xg(y)^* = 0$ for all $x, y \in R$. Define the new mappings $S, T\colon R \to Q_mr$ such that $S(x) = f(x)^*$ for all $x \in R$ and $T(x) = g(x)^*$ for all $x \in R$. Then S and T are additive, since f and g are additive. Therefore the last expression can be rewritten as $S(x)y + xT(y) = 0$ for all $x, y \in R$. In view of Lemma 2.5 in [27], we conclude that $S(x) = -xq$ for all $x \in R$ and $T(x) = qx$ for all $x \in R$, where $q \in Q_mr$. This further implies that $f(x)^* = -xq$ and $g(x)^* = qx$ for all $x \in R$. Taking involution on both sides we obtain $f(x) = -q^*x^*$ and $g(x) = x^*q^*$ for all $x \in R$. That is, $f(x) = -px^*$ and $g(x) = x^*p$ for all $x \in R$, where $p = q^* \in Q_mr$. This completes the proof of the proposition. □

3 (Jordan) $*$-centralizers on (semi)prime rings

Following [3], an additive mapping $T\colon R \to R$ is called a reverse left (resp. right) $*$-centralizer if $T(xy) = T(y)x^*$ (resp. $T(xy) = y^*T(x)$) holds for all $x, y \in R$. In Proposition 2.8 of Section 2.2, we proved that if there exists an additive mapping $T\colon R \to R$ where R is a 2-torsion free semiprime ring satisfying the relation $T(xyx) = x^*T(y)x^*$ for all $x, y \in R$, then T is a reverse $*$-centralizer. It is easy to see that any reverse $*$-centralizer T on an arbitrary ring R satisfies the relation

$$T(xyx) = x^*y^*T(x) - x^*T(y)x^* + T(x)y^*x^* \tag{1}$$

for all pairs $x, y \in R$. It seems natural to ask whether the relation (1) characterizes reverse $*$-centralizers among all additive mappings on 2-torsion free semiprime rings. The answer to this question is negative. Namely, a routine calculation shows that for any fixed element $a \in R$, where R is an arbitrary ring, the mapping $T\colon R \to R$ defined

by $T(x) = ax^* + x^*a$ for all $x \in R$ satisfies the relation (1). More precisely, we prove the following theorem.

Theorem 3.1. *Let R be a 2-torsion free semiprime ring with involution $'*'$. Suppose there exists an additive mapping $T\colon R \to R$ such that $T(xyx) = x^*y^*T(x) - x^*T(y)x^* + T(x)y^*x^*$ for all $x, y \in R$. Then there exists $p \in Q_S$ such that $2T(x) = px^* + x^*p$ for all $x \in R$.*

Proof. Suppose an additive mapping $T\colon R \to R$ satisfies the relation

$$T(xyx) = x^*y^*T(x) - x^*T(y)x^* + T(x)y^*x^* \tag{2}$$

for all $x, y \in R$. Invoking involution on both sides to (2), we get

$$T(xyx)^* = T(x)^*yx - xT(y)^*x + xyT(x) \tag{3}$$

for all $x, y \in R$. Define a new map $S\colon R \to R$ such that $S(x) = T(x)^*$ for all $x \in R$. Then (3) can be written as

$$S(xyx) = S(x)yx - xS(y)x + xyS(x)$$

for all $x, y \in R$. Further, since $T\colon R \to R$ is an additive mapping and so $S\colon R \to R$ is too. Thus in view of Theorem 2.1 in [27], we conclude that $2S(x) = qx + xq$ for all $x \in R$, where $q \in Q_S$ and hence $2T(x)^* = qx + xq$. This implies that $2T(x) = x^*q^* + q^*x^*$ for all $x \in R$. That is, $2T(x) = x^*p + px^*$ for all $x \in R$, where $p = q^* \in Q_S$. This completes the proof of the theorem. □

The next result is motivated by Theorem 4 in [24]

Theorem 3.2. *Let R be a prime ring with involution $'*'$ having a nonzero centre such that $char(R) \neq 6mn(m+n)$, $m \geq 1$, $n \geq 1$. Let $T\colon R \to R$ be an additive mapping such that $(m+n)T(x^2) = mx^*T(x) + nT(x)x^*$ for all $x \in R$, then $T(xy) = T(y)x^* = y^*T(x)$ for all $x, y \in R$.*

Proof. Consider a new map $S\colon R \to R$ defined by $S(x) = T(x)^*$ for all $x \in R$. Then it is easy to verify that S is additive. By the given hypothesis we have $(m+n)T(x^2) = mx^*T(x) + nT(x)x^*$ for all $x \in R$. Taking involution on both sides we get $(m+n)T(x^2)^* = mT(x)^*x + nxT(x)^*$ for all $x \in R$. That is, $(m+n)S(x^2) = mS(x)x + nxS(x)$ for all $x \in R$ where $S(x) = T(x)^*$. In view of Theorem 4 in [24], we are forced to conclude that $S(xy) = S(x)y = xS(y)$ for all $x, y \in R$. This gives $T(xy)^* = T(x)^*y = xT(y)^*$ for all $x, y \in R$. This yields that $T(xy) = y^*T(x) = T(y)x^*$ for all $x, y \in R$. This completes the proof. □

An immediate consequence of the above theorem is the following result.

Corollary 3.1. *Let R be a prime ring with involution $'*'$ having a nonzero centre such that $char(R) \neq 6mn(m+n)$, $m \geq 1$, $n \geq 1$. Let $T\colon R \to R$ be an additive mapping such that $(m+n)T(x^2) = mx^*T(x) + nT(x)x^*$ for all $x \in R$. Then there exists $q \in Q_S(R)$ such that $T(x) = qx^*$ for all $x \in R$.*

Proof. In view of Theorem 3.2 and Proposition 2.2 in [1], we get the required result. □

We conclude this section with the following theorem, which outlined at the begining of this paper.

Theorem 3.3. *Let R be a 2-torsion free semiprime ring with involution $'*'$. Then every Jordan $*$-centralizer of R is a reverse $*$-centralizer.*

Proof. By the assumption, we are given that $T: R \to R$ is a Jordan $*$-centralizer i.e.,

$$T(xy + yx) = T(x)y^* + y^* T(x) = x^* T(y) + T(y)x^* \tag{4}$$

for all $x, y \in R$. Substituting $xy + yx$ for y in (4), we obtain

$$T(x(xy + yx) + (xy + yx)x) = x^* T(xy + yx) + T(xy + yx)x^*$$

for all $x, y \in R$. Application of (4) forces that

$$T(x)(xy + yx)^* + (xy + yx)^* T(x) = x^* (T(x)y^* + y^* T(x)) + (T(x)y^* + y^* T(x))x^*$$

for all $x, y \in R$. This implies that

$$[T(x), x^*]y^* + y^* [x^*, T(x)] = 0$$

for all $x, y \in R$. This can be rewritten as

$$[T(x), x^*]y^* = y^* [T(x), x^*]$$

for all $x, y \in R$. In view of Lemma 2.4 we have $[T(x), x^*] \in Z(R)$ for all $x \in R$. We now prove that $[T(x), x^*] = 0$ for all $x \in R$. Take any $c \in Z$. Then for any $x \in R$, we have

$$\begin{aligned} 2T(cx) &= T(cx + xc) \\ &= T(c)x^* + x^* T(c) \\ &= 2T(c)x^* . \end{aligned}$$

Since R is 2-torsion free, we obtain $T(cx) = T(c)x^*$ for all $x \in R$. Also, we have

$$\begin{aligned} 2T(cx) &= T(c)x^* + x^* T(c) \\ &= c^* T(x) + T(x)c^* \\ &= 2T(x)c^* . \end{aligned}$$

This gives $T(cx) = T(x)c^*$ for all $x \in R$. Hence, we obtain $T(cx) = T(x)c^* = T(c)x^*$ for all $x \in R$ and $c \in Z(R)$. Thus, we have

$$\begin{aligned} [T(x), x^*]c^* &= T(x)x^* c^* - x^* T(x)c^* \\ &= T(x)c^* x^* - x^* T(x)c^* \\ &= T(c)(x^*)^2 - x^* T(c)x^* \\ &= T(c)(x^*)^2 - T(c)(x^*)^2 \\ &= 0 . \end{aligned}$$

Since $[T(x), x^*] \in Z(R)$ for all $x \in R$, so we have $[T(x), x^*] = 0$ for all $x \in R$. From relation (4), we find that $2T(x^2) = T(xx + xx) = T(x)x^* + x^*T(x) = 2T(x)x^* = 2x^*T(x)$ for all $x \in R$. Since R is 2-torsion free ring, the last expression yields that $T(x^2) = x^*T(x)$ for all $x \in R$. That is, T is Jordan left $*$-centralizer on R. In view of [3, Proposition 2.3], we conclude that T is a reverse $*$-centralizer of R. Thereby the proof is completed. □

A special case of the above theorem, which is of independent interest is the following corollary:

Corollary 3.2. *Let R be a 2-torsion free semiprime ring with involution $'*'$. Let T be the Jordan $*$-centralizer of R. Then $T(x) = qx^*$ for all $x \in R$, where $q \in Q_s(R)$.*

4 (Jordan)$*$-derivations in (semi)prime rings

Let R be a ring with involution $*$. An additive mapping $D\colon R \longrightarrow R$ is said to be a $*$-derivation if $D(xy) = D(x)y^* + xD(y)$ holds for all $x, y \in R$, and is called a Jordan $*$-derivation if $D(x^2) = D(x)x^* + xD(x)$ holds for all $x \in R$. The concept of Jordan $*$-derivations appears for the first time in [13]. The notion of Jordan $*$-derivations arise naturally in the theory of the representability of quadratic forms with sesquilinear functionals (see Šemrl's work [15] and [16]). More related results on Jordan $*$-derivations can be found in [4, 14] and [28].

A mapping f of a ring R into itself is called commuting (resp. skew commuting) on a subset S of R if $f(x)x - xf(x) = 0$ (resp. $f(x)x + xf(x) = 0$) holds for all $x \in S$. The study of such mappings was initiated a number of years ago by Posner [21]. In [21, Lemma 3], Posner proved that if a prime ring R admits a commuting derivation D on R, then R is commutative or $D = 0$. Over the last some decades, numerous papers concerning commuting and some related mappings have been published (see [9, 10, 11] and [12] for refrences). In [10], Brešar showed that if a prime ring R admits a nonzero derivation D such that $D(x)x-xD(x) \in Z(R)$ for all $x \in U$ or $D(x)x+xD(x) \in Z(R)$ for all $x \in U$, where U is a nonzero left ideal of R, then R is commutative. Further the above mentioned result was extended by Argac [5] as follows: Let R be a semiprime ring and D, G are derivations of R such that at least one is nonzero. If $D(x)x = xG(x)$ for all $x \in R$, then R has a nonzero central ideal. Moreover, if R is prime, then R is commutative. Our next aim is to study similar problems in the setting of rings with involution involving pair of Jordan $*$-derivations D and G of R. We begin with the following theorem.

Theorem 4.1. *Let R be a noncommutative prime ring with involution $'*'$. Suppose there exist Jordan $*$-derivations $D, G\colon R \to R$ such that $xD(y) \pm yG(x) = 0$ for all $x, y \in R$. Then $D = G = 0$.*

Proof. First we consider the case

$$xD(y) - yG(x) = 0 \tag{1}$$

for all $x, y \in R$. Replacing y by y^2 in (1), we get $0 = xD(y^2) - y^2G(x) = xD(y)y^* + xyD(y) - y^2G(x)$ for all $x, y \in R$. Using the given hypothesis we obtain $xD(y)y^* + xyD(y) - yxD(y) = 0$ for all $x, y \in R$. That is,

$$xD(y)y^* + [x, y]D(y) = 0 \tag{2}$$

for all $x, y \in R$. Replacing x by zx in (2), we get

$$zxD(y)y^* + z[x, y]D(y) + [z, y]xD(y) = 0 \tag{3}$$

for all $x, y, z \in R$. Left multiplying (2) by z, yields

$$zxD(y)y^* + z[x, y]D(y) = 0 \tag{4}$$

for all $x, y, z \in R$. Comparing (3) and (4), we obtain

$$[z, y]xD(y) = 0 \tag{5}$$

for all $x, y, z \in R$. Thus for each $y \in R$, by the primeness of R either $[z, y] = 0$ or $D(y) = 0$. Now, let $A = \{y \in R | [z, y] = 0 \text{ for all } z \in R\}$ and $B = \{y \in R | D(y) = 0\}$. Then A and B are additive subgroups of R and $R = A \cup B$. But a group can not be a union of two of its proper subgroups and hence either $R = A$ or $R = B$. Since we have assumed R to be noncommutative. So we are force to conclude that $R = B$. That is, $D(y) = 0$ for all $y \in R$. This intern implies $G = 0$ by (1).

The same argument can be adapted in the case $xD(y) + yG(x) = 0$ for all $x, y \in R$. This proves the theorem. □

Theorem 4.2. *Let R be a noncommutative prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Suppose there exist Jordan $*$-derivations $D, G\colon R \to R$ such that $xD(x) \pm xG(x) = 0$ for all $x \in R$. Then $D = G = 0$.*

Proof. First we consider the case

$$xD(x) - xG(x) = 0 \tag{6}$$

for all $x \in R$. Linearizing (6), we get

$$yD(x) + xD(y) = yG(x) - xD(y)$$

for all $x, y \in R$. This can be further written as

$$y(D - G)(x) - x(G - D)(y) = 0$$

for all $x, y \in R$. Now if D and G are Jordan $*$-derivations on R, then $D - G$ and $G - D$ are also Jordan $*$-derivations on R. Therefore in view of Theorem 4.4, we conclude that $D - G = 0$ and $G - D = 0$. That is, $D = G$. In view of (6), we get $[D(x), x] = 0$ for all $x \in R$ and $[G(x), x] = 0$ for all $x \in R$. Thus by Lemma 2.2, we obtain $D = 0$ and $G = 0$.

The same argument can be adapted in the case $xD(x) + xG(x) = 0$ for all $x \in R$. □

Theorem 4.3. *Let R be a noncommutative prime ring with involution $'*'$ of the second kind such that $char(R) \neq 2$. Suppose there exist Jordan $*$-derivations $D, G\colon R \to R$ such that $D(x)y \pm yG(x) = 0$ for all $x, y \in R$. Then $D = G = 0$.*

Proof. First we consider the case

$$D(x)y - yG(x) = 0 \tag{7}$$

for all $x, y \in R$. Replacing y by yz in (7), we get

$$D(x)yz - yzG(x) = 0 \tag{8}$$

for all $x, y, z \in R$. Right multiplying (7) by z, we obtain

$$D(x)yz - yG(x)z = 0 \tag{9}$$

for all $x, y, z \in R$. Comparing (8) and (9), we get $y[G(x), z] = 0$ for all $x, y, z \in R$. This further implies $[G(x), z]y[G(x), z] = 0$ for all $x, y, z \in R$. Hence, the primeness of R forces that

$$[G(x), z] = 0 \tag{10}$$

for all $x, z \in R$. Replacing x by x^2 in (10), and using the fact G is a Jordan $*$-derivation, we get $0 = [G(x^2), z] = [G(x)x^* + xG(x), z] = G(x)[x^*, z] + [x, z]G(x)$. This further implies that $G(x)[x+x^*, z] = 0$ for all $x, z \in R$. Replacing z by zy in the last expression, we obtain $G(x)z[x+x^*, y] = 0$ for all $x, y, z \in R$. Thus for each $x \in R$, by the primeness of R either $G(x) = 0$ or $[x + x^*, y] = 0$. Now let $A = \{x \in R | G(x) = 0\}$ and $B = \{x \in R | [x+x^*, y] = 0 \text{ for all } y \in R\}$. Thus A and B are additive subgroups of R and $R = A \cup B$. But a group can not be a union of two of its proper subgroups and hence either $R = A$ or $R = B$. Suppose $R = B$, then $[x+x^*, y] = 0$ for all $x, y \in R$. Replacing x by $h+k$, where $h \in H(R)$ and $k \in S(R)$, we get $2[h, y] = 0$. Since $char(R) \neq 2$, we obtain $[h, y] = 0$ for all $h \in H(R)$ and $y \in R$. That is, $H(R) \subseteq Z(R)$. Thus R is commutative by Lemma 2.3. Which gives a contradiction. Therefore we must have $R = A$. That is, $G = 0$. This intern implies that $D = 0$ in view of equation (7).

The same argument can be adapted in the case $D(x)y + yG(x) = 0$ for all $x, y \in R$. Thereby completing the proof of the theorem. □

Theorem 4.4. *Let m, n be fixed non-negative integers, and R be a $(m + n)!$-torsion free noncommutative prime ring with involution $'*'$ of the second kind having the identity element e. Suppose there exist Jordan $*$-derivations $D, G\colon R \to R$ such that $D(x^m)x^n \pm x^nG(x^m) = 0$ for all $x \in R$. Then $D = G = 0$.*

Proof. First we assume that

$$D(x^m)x^n - x^nG(x^m) = 0 \tag{11}$$

for all $x \in R$. Substituting $x + \lambda y$ for x in (11), we obtain

$$\lambda P_1(x, y) + \lambda^2 P_2(x, y) + \cdots + \lambda^{(m+n)} P_{(m+n)}(x, y) = 0$$

for all $x, y \in R$, where $\lambda \in Z$ and $P_i(x, y)$ denotes the sum of terms involving i factors of y in the expansion of $D((x + \lambda y)^m)(x + \lambda y)^n - (x + \lambda y)^n G((x + \lambda y)^m) = 0$. In view of Lemma 1 in [17], we obtain

$$\begin{aligned} P_1(x, y) = {} & D(x^{m-1}y + x^{m-2}yx + \cdots + yx^{m-1})x^n \\ & + D(x^m)(x^{n-1}y + x^{n-2}yx + \cdots + yx^{n-1}) \\ & - (x^{n-1}y + x^{n-2}yx + \cdots + yx^{n-1})G(x^m) \\ & - x^n G(x^{m-1}y + x^{m-2}yx + \cdots + yx^{m-1}) \end{aligned} \tag{12}$$

for all $x, y \in R$. Taking $x = e$ into (12) and noting that $D(e) = 0$ and $G(e) = 0$, we obtain

$$mD(y) - mG(y) = 0 \tag{13}$$

for all $y \in R$. Since R is $(m + n)!$-torsion free, we obtain

$$D(y) - G(y) = 0 \tag{14}$$

for all $y \in R$. Using a similar computational way to (12), we also have

$$P_2(e, y) = nmD(y)y + \frac{(m+1)m}{2} D(y^2) - nmyG(y) - \frac{(m+1)m}{2} G(y^2) \tag{15}$$

for all $y \in R$. Application of (14) yields that

$$nmD(y)y - nmyD(y) = 0$$

for all $y \in R$. Since R is $(m + n)!$-torsion free, we obtain $[D(y), y] = 0$ for all $y \in R$. Further in view of expression (14), we conclude that $[G(y), y] = 0$ for all $y \in R$. Thus in view of Lemma 2.2, we obtain $D = 0$ and $G = 0$.

By the similar approach, we obtain the same conclusion in case $D(x^m)x^n + x^n G(x^m) = 0$ for all $x \in R$. This proves the theorem completely. □

Corollary 4.1. *Let R be a noncommutative prime ring with involution $'*'$ of the second kind and of characteristic different form two, having the identity element e. Suppose there exist Jordan $*$-derivations $D, G\colon R \to R$ such that $D(x)x \pm xG(x) = 0$ for all $x \in R$. Then $D = G = 0$.*

We conclude the paper with the following example which demonstrates that it is essential for R to be prime in the hypothesis of Theorems 4.1, 4.2 and 4.3.

Example 2. *Let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b, c \in Z_2 \right\}$. *Then R is a noncommutative ring under usual matrix operations. Define mappings $D\colon R \longrightarrow R$, $G\colon R \longrightarrow R$,*

and $*: R \longrightarrow R$ *as follows* $D\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$.

Then, it is easy to verify that D, G are Jordan $*$-derivations on R, and satisfy all the requirements of Theorem 4.1, 4.2 and 4.3. However, neither $D = 0$ nor $G = 0$.

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Mohammad Ashraf and Ghulam Mohammad

Skew constacyclic codes over $F_q + vF_q + v^2F_q$

Abstract: In the present paper, we study $(1-2v^2)$-skew constacyclic codes over the ring $F_q + vF_q + v^2F_q$, where $v^3 = v$, $q = p^m$ and p is an odd prime. We investigate the structural properties of skew cyclic codes over $F_q + vF_q + v^2F_q$ using decomposition method. By defining a Gray map from $F_q + vF_q + v^2F_q$ to F_q^3, it has been proved that the Gray image of a $(1-2v^2)$-skew constacyclic code of length n over $F_q + vF_q + v^2F_q$ is a skew cyclic code of length $2n$ over F_q. Finally, the idempotent generators of $(1-2v^2)$-skew constacyclic codes over $F_q + vF_q + v^2F_q$ have also been studied.

Keywords: Gray map; Skew polynomial rings; Skew constacyclic codes; Idempotent generators.

1 Introduction

During the last decade of the twentieth century a great deal of attention has been given to the study of linear codes over finite rings because of their new role in algebraic coding theory and their successful applications. The class of cyclic codes is a very important class of linear codes from both theoretical and practical point of view which are easier to implement due to their rich algebraic structure. Cyclic codes have been studied for the last six decades. Based on these facts, cyclic codes have become one of the most important class in coding theory. A landmark paper by Hammons, et al. [16] discovered that some good nonlinear codes over $\mathbb{Z}_2$ can be viewed as binary images under a Gray map of linear cyclic codes over $\mathbb{Z}_4$. But all this work is restricted to codes that are defined in a commutative ring.

Boucher et al. [8, 9] and [10] studied the structure of skew cyclic codes over a non commutative ring $F[x, \theta]$, called skew polynomial ring, where F is a finite field and θ is a field automorphism of F. They generalized the class of linear and cyclic codes to the class of skew cyclic codes by using the ring $F[x, \theta]$, where the generator polynomials of skew cyclic codes come from the ring $F[x, \theta]$. They also gave some examples of skew cyclic codes with Hamming distances larger than the best known linear codes with the same parameters. Later on, Abualrub et al. [2] and Bhaintwal [7], defined skew quasi cyclic codes over these classes of rings. The main motivation of studying codes in this setting is that polynomials in skew polynomial rings exhibit many factorizations and hence there are many ideals in skew polynomial ring than in the commutative ring. But

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all this work is restricted to the condition that the order of the automorphism must be a factor of the length of the code. In [17], Siap, et al. removed this condition and they studied the structural properties of skew cyclic codes of arbitrary length over finite fields. A lot of work has been done in this direction (see references [1, 4, 11]).

Jitman et al. [14] defined skew constacyclic codes by defining the skew polynomial ring with coefficients from finite chain rings, especially the ring $F_{p^m} + uF_{p^m}$ where $u^2 = 0$. Further Gursoy et al. [13] investigated the structural properties of skew cyclic codes through the decomposition method over $F_q + vF_q$, where $v^2 = v$ and $q = p^m$. Recently, the authors [4] studied the structural properties of skew cyclic codes over the ring $F_3 + vF_3$ with $v^2 = 1$ by considering the automorphism as; $\theta : v \mapsto -v$. They proved that skew cyclic codes over $F_3 + vF_3$ are equivalent to either cyclic codes or quasi cyclic codes. Further, they studied skew cyclic codes over the ring $F_q + vF_q$ with $v^2 = 1$ in [5] by using decomposition method. Very recently, AL-Ashker and Abu-Jafar [3] investigated the structural properties of skew constacyclic codes over the ring $F_p + vF_p$ with $v^2 = v$. Motivated by the study of AL-Ashker and Abu-Jafar [3], in the present paper, we study $(1-2v^2)$-skew constacyclic codes over the ring $F_q+vF_q+v^2F_q$, where $v^3 = v,\ q = p^m$ and p is an odd prime.

Throughout the paper R will denote the ring $F_q + vF_q + v^2F_q$ with $v^3 = v,\ q = p^m$ and p is an odd prime. Consider the automorphism $\theta_t\colon R \longrightarrow R$ such that $\theta_t(a + vb + v^2c) = a^{p^t} + vb^{p^t} + v^2c^{p^t}$. It is to be noted that θ_1 is the Frobenius automorphism of F_q and $\theta_t = \theta_1^t$.

2 Preliminaries

Let $R = F_q + vF_q + v^2F_q$, where $q = p^m$ and p is an odd prime. R is a commutative and non-chain ring with characteristic p which contains q^3 elements. The ring is endowed with the natural addition and multiplication with the property $v^3 = v$ and it can be viewed as the quotient ring $F_q[v]/\langle v^3 - v\rangle$. The elements of R can be uniquely written as $a+vb+v^2c$, where $a,\ b,\ c \in F_q$. It is a semi-local ring having three maximal ideals $\langle v\rangle,\ \langle v-1\rangle$ and $\langle v+1\rangle$.

Define a mapping $\theta_t\colon R \longrightarrow R$ such that $\theta_t(a + vb + v^2c) = a^{p^t} + vb^{p^t} + v^2c^{p^t}$ for all $a,\ b,\ c \in F_q$. One can verify that θ_t is an automorphism on R and $\theta_t = \theta_1^t$. This automorphism acts on F_q as follows:

$$\begin{aligned}\theta_t\colon F_q &\longrightarrow F_q\\ a &\mapsto a^{p^t}.\end{aligned}$$

It may be noted that the order of this automorphism is $|\langle\theta_t\rangle| = m/t$ and the subring $F_{p^t} + vF_{p^t} + v^2F_{p^t}$ of R is invariant under θ_t.

Definition 2.1. *For a given automorphism θ_t of R, the set $R[x,\theta_t] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in R, n \geq 0\}$ of formal polynomials forms a ring under usual addition of*

polynomials and multiplication is defined by the rule $(ax^i)(bx^j) = a\theta_t^i(b)x^{i+j}$. *The ring* $R[x, \theta_t]$ *is called skew polynomial ring over* R.

It can be easily seen that the ring $R[x, \theta_t]$ is non-commutative unless θ_t is the identity automorphism on R. Therefore, when an ideal of $R[x, \theta_t]$ is considered, one should specify whether it is a right ideal or a left ideal. The skew polynomial ring $R[x, \theta_t]$ is not left or right Euclidean. However, the division algorithm holds for some polynomials whose leading coefficients are invertible (for detail see references [9] and [14]).

3 Gray map and linear codes over *R*

Gao [12], initiated the study of linear codes over the ring $F_p + uF_p + u^2F_p$, where $u^3 = u$ and p is an odd prime. Later on, Mostafanasab and Karimi [15], studied $(1 - 2v^2)$-constacyclic codes over the ring $F_p + uF_p + u^2F_p$, where $u^3 = u$ and p is an odd prime. Let R^n be the set of all n-tuples over the ring R. Then any nonempty subset C of R^n is called a code of length n over R. C is called linear code of length n over R if it is an R-submodule of R^n. Elements of C are called codewords and therefore each codeword c in such a code C is just an n-tuple of the form $c = (c_0, c_1, \cdots, c_{n-1}) \in R^n$.

Let C be a linear code of length n over R. Then C is said to be cyclic if for every $(c_0, c_1, \cdots, c_{n-1}) \in C$ implies that $(c_{n-1}, c_0, \cdots, c_{n-2}) \in C$, negacyclic if $(c_0, c_1, \cdots, c_{n-1}) \in C$ implies that $(-c_{n-1}, c_0, \cdots, c_{n-2}) \in C$ and $(1 - 2v^2)$- constacyclic if $(c_0, c_1, \cdots, c_{n-1}) \in C$ implies that $((1 - 2v^2)c_{n-1}, c_0, \cdots, c_{n-2}) \in C$.

The Hamming weight $w_H(r)$ of a codeword $r = (r_0, r_1, \cdots, r_{n-1}) \in R^n$ is the number of nonzero components. The minimum weight $w_H(C)$ of a code C is the smallest weight among all its nonzero codewords. For $r = (r_0, r_1, \cdots, r_{n-1})$, $s = (s_0, s_1, \cdots, s_{n-1}) \in R^n$, $d_H(r, s) = |\{i \mid r_i \neq s_i\}|$ is called the Hamming distance between r and $s \in R^n$ and is denoted by

$$d_H(r, s) = w_H(r - s) .$$

The minimum Hamming distance between distinct pairs of codewords of a code C is called the minimum distance of C and is denoted by $d_H(C)$ or shortly d_H.

Now, we define the Lee weight of an element $r = a + vb + v^2c \in R$ as follows:

$$w_L(r) = w_H(-c, 2a + c) ,$$

where w_H denotes the usual Hamming weight on F_q. Let $r = (r_0, r_1, \cdots, r_{n-1})$ be a vector in R^n. Then the Lee weight of r is the rational sum of Lee weights of its components, that is, $w_L(r) = \sum_{i=0}^{n-1} w_L(r_i)$. For any two elements $r, s \in R^n$, the Lee distance is given by $d_L(r, s) = w_L(r - s)$. The minimum Lee distance of a code C is the smallest nonzero Lee distance between all pairs of distinct codewords. The minimum Lee weight of C is the

smallest nonzero Lee weight among all codewords. If C is linear, then the minimum Lee distance is the same as the minimum Lee weight.

The Gray map ϕ from R to F_q^2 is defined as $\phi(a + vb + v^2c) = (-c, 2a + c)$. It can be easily seen that ϕ is linear. The Gray map ϕ can be extended to R^n in a natural way, that is, $\phi\colon R^n \longrightarrow F_q^{2n}$ such that $\phi(r_0, r_1, \cdots, r_{n-1}) = (-c_0, -c_1, \cdots, -c_{n-1}, 2a_0 + c_0, 2a_1 + c_1 \cdots, 2a_{n-1} + c_{n-1})$, where $r_i = a_i + vb_i + v^2c_i$ for $i = 0, 1, \cdots, n-1$. The map ϕ is a F_q-linear map but not one to one.

For a code C over R, define

$$C_1 = \{x \in F_q^n \mid (1 - v^2)x + \frac{p+1}{2}(v^2 + v)y + \frac{p+1}{2}(v^2 - v)z \in C \text{ some } y, z \in F_q^n\},$$

$$C_2 = \{y \in F_q^n \mid (1 - v^2)x + \frac{p+1}{2}(v^2 + v)y + \frac{p+1}{2}(v^2 - v)z \in C \text{ some } x, z \in F_q^n\},$$

and

$$C_3 = \{z \in F_q^n \mid (1 - v^2)x + \frac{p+1}{2}(v^2 + v)y + \frac{p+1}{2}(v^2 - v)z \in C \text{ some } x, y \in F_q^n\}.$$

If C is linear code of length n over R, then $C_1,\ C_2$ and C_3 are all linear codes of length n over F_q. Moreover, the linear code C of length n over R can be uniquely expressed as

$$C = (1 - v^2)C_1 \oplus \frac{p+1}{2}(v^2 + v)C_2 \oplus \frac{p+1}{2}(v^2 - v)C_3$$

and $|C| = |C_1||C_2||C_3|$.

A generator matrix of C is a matrix whose rows generate C. Let

$$C = (1 - v^2)C_1 \oplus \frac{p+1}{2}(v^2 + v)C_2 \oplus \frac{p+1}{2}(v^2 - v)C_3$$

be a linear code of length n over R with generator matrix G. Then G can be written as

$$G = \begin{pmatrix} (1 - v^2)G_1 \\ \\ \frac{p+1}{2}(v^2 + v)G_2 \\ \\ \frac{p+1}{2}(v^2 - v)G_3 \end{pmatrix},$$

where $G_1,\ G_2$ and G_3 are the generator matrices of $C_1,\ C_2$ and C_3 respectively.

Let $r = (r_0, r_1, \cdots, r_{n-1})$ and $s = (s_0, s_1, \cdots, s_{n-1})$ be two elements of R^n. Then the Euclidean inner product of r and s in R^n is defined as

$$r \cdot s = r_0s_0 + r_1s_1 + \cdots + r_{n-1}s_{n-1}.$$

The dual code $C^\perp$ of C is defined as

$$C^\perp = \{r \in R^n \mid r \cdot c = 0,\ \text{for all } c \in C\}.$$

A code C is called self-orthogonal if $C \subseteq C^\perp$ and self dual if $C = C^\perp$.

One of the properties of the Gray map defined above is that it preserves the self-orthogonality as given in the following lemma:

Lemma 3.1 ([15, Proposition 2]). *Let C be a code of length n over R such that $C \subset (F_q + vF_q + v^2F_q)^n$. If C is self-orthogonal, then so is $\phi(C)$.*

Theorem 3.1 ([15, Theorem 8]). *Let $C = (1-v^2)C_1 \oplus \frac{p+1}{2}(v^2+v)C_2 \oplus \frac{p+1}{2}(v^2-v)C_3$ be a code of length n over R. Then C is a $(1-2v^2)$-constacyclic code if and only if $C^\perp$ is also a $(1-2v^2)$-constacyclic code and $C^\perp = (1-v^2)C_1^\perp \oplus \frac{p+1}{2}(v^2+v)C_2^\perp \oplus \frac{p+1}{2}(v^2-v)C_3^\perp$.*

4 $(1-2v^2)$-Skew constacyclic codes over R

Skew cyclic codes over the ring R were studied by the authors [6]. In the present section, we generalise this study to $(1-2v^2)$-skew constacyclic codes over R. Let θ_t be an automorphism on R given by $\theta_t(a + vb + v^2c) = a^{p^t} + vb^{p^t} + v^2c^{p^t}$. Then a linear code C of length n over R is called a skew cyclic code or θ_t-cyclic code if for each $c = (c_0, c_1, \cdots, c_{n-1}) \in C$ implies that $\sigma(c) = (\theta_t(c_{n-1}), \theta_t(c_0), \cdots, \theta_t(c_{n-2})) \in C$, where $\sigma(c)$ denotes the skew cyclic shift of c, C is called skew negacyclic code if for each $c = (c_0, c_1, \cdots, c_{n-1}) \in C$ implies that $\nu(c) = (-\theta_t(c_{n-1}), \theta_t(c_0), \cdots, \theta_t(c_{n-2})) \in C$, where $\nu(c)$ denotes the skew negacyclic shift of c and C is called $(1-2v^2)$-skew constacyclic code if for each $c = (c_0, c_1, \cdots, c_{n-1}) \in C$ implies that $\tau(c) = ((1-2v^2)\theta_t(c_{n-1}), \theta_t(c_0), \cdots, \theta_t(c_{n-2})) \in C$, where $\tau(c)$ denotes the skew constacyclic shift of c.

Now, consider $R[x, \theta_t]/\langle x^n - (1-2v^2)\rangle$. It can be easily seen that $R[x, \theta_t]/\langle x^n - (1-2v^2)\rangle$ is a left $R[x, \theta_t]$ module under the following operations:

$$(f(x) + \langle x^n - (1-2v^2)\rangle) + (g(x) + \langle x^n - (1-2v^2)\rangle) = (f(x) + g(x)) + \langle x^n - (1-2v^2)\rangle\,,$$

$$r(x)(f(x) + \langle x^n - (1-2v^2)\rangle)) = r(x)f(x) + \langle x^n - (1-2v^2)\rangle$$

for any $r(x) \in R[x, \theta_t]$. By the definition of $R[x, \theta_t]/\langle x^n - (1-2v^2)\rangle$, we can identify each codeword $c = (c_0, c_1, \cdots, c_{n-1})$ of the $(1-2v^2)$-skew constacyclic code C by a polynomial $c(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1}$.

In the following lemma, we give a module structure of $(1-2v^2)$-skew constacyclic codes of arbitrary length.

Lemma 4.1. *A code C of length n over R is a $(1-2v^2)$-skew constacyclic code if and only if C is a left $R[x, \theta_t]$-submodule of $R[x, \theta_t]/\langle x^n - (1-2v^2)\rangle$.*

Now, we give the characterisation of $(1-2v^2)$-skew constacyclic codes over R as follows:

Theorem 4.1. *Let $C = (1-v^2)C_1 \oplus \frac{p+1}{2}(v^2+v)C_2 \oplus \frac{p+1}{2}(v^2-v)C_3$ be a linear code of length n over R. Then C is a $(1-2v^2)$-skew constacyclic code over R with respect to automorphism θ_t if and only if C_1 is a skew cyclic code and C_2, C_3 are skew negacyclic codes of length n over F_q respectively with respect to the same automorphism θ_t.*

Proof. For any $r = (r_0, r_1, \cdots, r_{n-1}) \in C$, we can write its components as $r_i = (1-v^2)a_i + \frac{p+1}{2}(v^2+v)b_i + \frac{p+1}{2}(v^2-v)c_i$, where $a_i,\ b_i,\ c_i \in F_q$, $0 \le i \le n-1$. Let $a = (a_0, a_1, \cdots, a_{n-1})$, $b = (b_0, b_1, \cdots, b_{n-1})$ and $c = (c_0, c_1, \cdots, c_{n-1})$. Then $a \in C_1$, $b \in C_2$ and $c \in C_3$. Now, suppose C_1 is a skew cyclic code and $C_2,\ C_3$ are skew negacyclic codes over F_q respectively with respect to the automorphism θ_t. This means that $\sigma(a) = (\theta_t(a_{n-1}), \theta_t(a_0), \cdots, \theta_t(a_{n-2})) = (a_{n-1}^{p^t}, a_0^{p^t}, \cdots, a_{n-2}^{p^t}) \in C_1$, $\nu(b) = (-\theta_t(b_{n-1}), \theta_t(b_0), \cdots, \theta_t(b_{n-2})) = (-b_{n-1}^{p^t}, b_0^{p^t}, \cdots, b_{n-2}^{p^t}) \in C_2$ and $\nu(c) = (-\theta_t(c_{n-1}), \theta_t(c_0), \cdots, \theta_t(c_{n-2})) = (-c_{n-1}^{p^t}, c_0^{p^t}, \cdots, c_{n-2}^{p^t}) \in C_3$. Thus $(1-v^2)\sigma(a) + (v^2+v)\frac{p+1}{2}\nu(b) + (v^2-v)\frac{p+1}{2}\nu(c) \in C$. It can be easily seen that

$$(1-v^2)\sigma(a) + (v^2+v)\frac{p+1}{2}\nu(b) + (v^2-v)\frac{p+1}{2}\nu(c) = \tau(r)\,.$$

Hence $\tau(r) \in C$, which means that C is a $(1-2v^2)$-skew constacyclic code over R with respect to the automorphism θ_t.

Conversely, suppose that C is a $(1-2v^2)$-skew constacyclic code over R with respect to the automorphism θ_t. Let $r_i = (1-v^2)a_i + \frac{p+1}{2}(v^2+v)b_i + \frac{p+1}{2}(v^2-v)c_i$, for any $a = (a_0, a_1, \cdots, a_{n-1}) \in C_1$, $b = (b_0, b_1, \cdots, b_{n-1}) \in C_2$ and $c = (c_0, c_1, \cdots, c_{n-1}) \in C_3$. Then $r = (r_0, r_1, \ldots, r_{n-1}) \in C$. By the hypothesis $\tau(r) \in C$. Since $(1-v^2)\sigma(a)+(v^2+v)\frac{p+1}{2}\nu(b)+(v^2-v)\frac{p+1}{2}\nu(c) = \tau(r)$, we get $(1-v^2)\sigma(a)+(v^2+v)\frac{p+1}{2}\nu(b)+(v^2-v)\frac{p+1}{2}\nu(c) \in C$. Thus $\sigma(a) \in C_1$, $\nu(b) \in C_2$ and $\nu(c) \in C_3$, which implies that C_1 is a skew cyclic code and $C_2,\ C_3$ are skew negacyclic codes of length n over F_q with respect to the automorphism θ_t. □

Corollary 4.1. *Let C be a $(1-2v^2)$-skew constacyclic code of length n over R. Then the dual code $C^\perp$ is also a $(1-2v^2)$-skew constacyclic code of length n over R.*

Proof. In view of Theorem 3.1, we know that $C^\perp = (1-v^2)C_1^\perp \oplus \frac{p+1}{2}(v^2+v)C_2^\perp \oplus \frac{p+1}{2}(v^2-v)C_3^\perp$. Since the dual code of every skew constacyclic code over F_q is also skew constacyclic ([9, 10]), by Theorem 4.1, $C^\perp$ is a $(1-2v^2)$-skew constacyclic code over R. □

Lemma 4.2. *Let τ be the $(1-2v^2)$-skew constacyclic shift of R^n and σ be the skew cyclic shift of F_q^{2n}. If ϕ is the Gray map of R^n into F_q^{2n}, then $\phi\tau = \sigma\phi$.*

Proof. Let $r = (r_0, r_1, \cdots, r_{n-1}) \in R^n$, where $r_i = a_i + vb_i + v^2c_i$ with $a_i,\ b_i,\ c_i \in F_q$ for $0 \le i \le n-1$. Taking $(1-2v^2)$-skew constacyclic shift of r, we have

$$\begin{aligned}
\tau(r) &= ((1-2v^2)\theta_t(r_{n-1}), \theta_t(r_0), \cdots, \theta_t(r_{n-2}))\\
&= ((1-2v^2)(a_{n-1}^{p^t} + vb_{n-1}^{p^t} + v^2c_{n-1}^{p^t}), a_0^{p^t} + vb_0^{p^t} + v^2c_0^{p^t}, \cdots, a_{n-2}^{p^t}\\
&\quad +vb_{n-2}^{p^t} + v^2c_{n-2}^{p^t})\\
&= (a_{n-1}^{p^t} - vb_{n-1}^{p^t} + v^2(-2a_{n-1}^{p^t} - c_{n-1}^{p^t}), a_0^{p^t} + vb_0^{p^t} + v^2c_0^{p^t}, \cdots, a_{n-2}^{p^t}\\
&\quad +vb_{n-2}^{p^t} + v^2c_{n-2}^{p^t})\,.
\end{aligned}$$

Now, using the definition of Gray map ϕ, we can deduce that

$$\phi(\tau(r)) = (2a_{n-1}^{p^t} + c_{n-1}^{p^t}, -c_0^{p^t}, \cdots, -c_{n-2}^{p^t}, -c_{n-1}^{p^t}, 2a_0^{p^t} + c_0^{p^t}, \cdots, 2a_{n-2}^{p^t} + c_{n-2}^{p^t})\,.$$

On the other hand, $\phi(r) = (-c_0, -c_1, \cdots, -c_{n-1}, 2a_0 + c_0, 2a_1 + c_1, \cdots, 2a_{n-1} + c_{n-1})$. Hence, $\sigma(\phi(r)) = (2a_{n-1}^{p^t} + c_{n-1}^{p^t}, -c_0^{p^t}, -c_1^{p^t}, \cdots, -c_{n-1}^{p^t}, -c_{n-2}^{p^t}, 2a_0^{p^t} + c_0^{p^t}, 2a_1^{p^t} + c_1^{p^t}, \cdots, 2a_{n-2}^{p^t} + c_{n-2}^{p^t})$. Therefore,

$$\phi\tau = \sigma\phi . \qquad \square$$

As a consequence of Lemma 4.2, we get the following main result:

Theorem 4.2. *Let C be a code of length n over R. Then C is* $(1-2v^2)$*-skew constacyclic code if and only if* $\phi(C)$ *is a skew cyclic code of length* $2n$ *over* F_q*.*

The following theorem gives the generators polynomials of $(1-2v^2)$-skew constacyclic codes over R:

Theorem 4.3. *Let* $C = (1-v^2)C_1 \oplus \frac{p+1}{2}(v^2+v)C_2 \oplus \frac{p+1}{2}(v^2-v)C_3$ *be* $(1-2v^2)$*-skew constacyclic code of length n over R. Then*

$$C = \langle (1-v^2)g_1(x), \frac{p+1}{2}(v^2+v)g_2(x), \frac{p+1}{2}(v^2-v)g_3(x)\rangle$$

and $|C| = q^{3n-deg(g_1(x))-deg(g_2(x))-deg(g_3(x))}$, *where* $g_1(x),\ g_2(x)$ *and* $g_3(x)$ *are the generator polynomials of* $C_1,\ C_2$ *and* C_3 *respectively.*

Proof. Since $C_1 = \langle g_1(x)\rangle \subseteq F_q[x,\theta_t]/\langle x^n-1\rangle,\ C_2 = \langle g_2(x)\rangle \subseteq F_q[x,\theta_t]/\langle x^n+1\rangle$, $C_3 = \langle g_3(x)\rangle \subseteq F_q[x,\theta_t]/\langle x^n+1\rangle$ and $C = (1-v^2)C_1 \oplus \frac{p+1}{2}(v^2+v)C_2 \oplus \frac{p+1}{2}(v^2-v)C_3$, we find that $C = \{c(x) \mid c(x) = (1-v^2)f_1(x) + \frac{p+1}{2}(v^2+v)f_2(x) + \frac{p+1}{2}(v^2-v)f_3(x),\ f_1(x) \in C_1,\ f_2(x) \in C_2,\ f_3(x) \in C_3\}$. Therefore

$$C \subseteq \langle (1-v^2)g_1(x), \frac{p+1}{2}(v^2+v)g_2(x), \frac{p+1}{2}(v^2-v)g_3(x)\rangle \subseteq R[x,\theta_t]/\langle x^n-(1-2v^2)\rangle .$$

For any $(1-v^2)k_1(x)g_1(x) + \frac{p+1}{2}(v^2+v)k_2(x)g_2(x) + \frac{p+1}{2}(v^2-v)k_3(x)g_3(x) \in \langle (1-v^2)g_1(x), \frac{p+1}{2}(v^2+v)g_2(x), \frac{p+1}{2}(v^2-v)g_3(x)\rangle \subseteq R[x,\theta_t]/\langle x^n-(1-2v^2)\rangle$, where $k_1(x) \in R[x,\theta_t]/\langle x^n-1\rangle$, $k_2(x), k_3(x) \in R[x,\theta_t]/\langle x^n+1\rangle$, there are $r_1(x), r_2(x), r_3(x) \in F_q[x,\theta_t]$ such that

$$(1-v^2)k_1(x) = (1-v^2)r_1(x) ,$$

$$\frac{p+1}{2}(v^2+v)k_2(x) = \frac{p+1}{2}(v^2+v)r_2(x)$$

and

$$\frac{p+1}{2}(v^2-v)k_3(x) = \frac{p+1}{2}(v^2-v)r_3(x) .$$

This means that

$$\langle (1-v^2)g_1(x), \frac{p+1}{2}(v^2+v)g_2(x), \frac{p+1}{2}(v^2-v)g_3(x)\rangle \subseteq C .$$

Hence $\langle (1-v^2)g_1(x), \frac{p+1}{2}(v^2+v)g_2(x), \frac{p+1}{2}(v^2-v)g_3(x)\rangle = C$. Since $|C| = |C_1||C_2||C_3|$, $|C| = q^{3n-deg(g_1(x))-deg(g_2(x))-deg(g_3(x))}$. $\square$

Theorem 4.4. *Let C_1 be skew cyclic code over F_q and C_2, C_3 be skew negacyclic codes over F_q with monic generator polynomials $g_1(x)$, $g_2(x)$ and $g_3(x)$ respectively. If $C = (1-v^2)C_1 \oplus \frac{p+1}{2}(v^2+v)C_2 \oplus \frac{p+1}{2}(v^2-v)C_3$ is a $(1-2v^2)$-skew constacyclic code of length n over R, then there is a unique polynomial $g(x) \in R[x,\theta_t]$ such that $C = \langle g(x)\rangle$ and $g(x)$ is a right divisor of $x^n - (1-2v^2)$, where $g(x) = (1-v^2)g_1(x) + \frac{p+1}{2}(v^2+v)g_2(x) + \frac{p+1}{2}(v^2-v)g_3(x)$.*

Proof. By Theorem 4.3, we may assume that

$$C = \left\langle (1-v^2)g_1(x), \frac{p+1}{2}(v^2+v)g_2(x), \frac{p+1}{2}(v^2-v)g_3(x)\right\rangle ,$$

where $g_1(x)$, $g_2(x)$ and $g_3(x)$ are the monic generator polynomials of C_1, C_2 and C_3 respectively. Let $g(x) = (1-v^2)g_1(x) + \frac{p+1}{2}(v^2+v)g_2(x) + \frac{p+1}{2}(v^2-v)g_3(x)$. Clearly, $\langle g(x)\rangle \subseteq C$. Note that

$$(1-v^2)g_1(x) = (1-v^2)g(x) ,$$

$$\frac{p+1}{2}(v^2+v)g_2(x) = \frac{p+1}{2}(v^2+v)g(x)$$

and

$$\frac{p+1}{2}(v^2-v)g_3(x) = \frac{p+1}{2}(v^2-v)g(x) ,$$

so $C \subseteq \langle g(x)\rangle$. Hence $C = \langle g(x)\rangle$. Since $g_1(x)$ is a monic right divisor of $x^n - 1$ and $g_2(x)$, $g_3(x)$ are monic right divisors of $x^n + 1$, there are $r_1(x) \in F_q[x,\theta_t]/\langle x^n - 1\rangle$ and $r_2(x), r_3(x) \in F_q[x,\theta_t]/\langle x^n + 1\rangle$ such that

$$x^n - 1 = r_1(x)g_1(x),\ x^n + 1 = r_2(x)g_2(x) = r_3(x)g_3(x) .$$

This implies that

$$x^n - (1-2v^2) = \left[(1-v^2)r_1(x) + \frac{p+1}{2}(v^2+v)r_2(x) + \frac{p+1}{2}(v^2-v)r_3(x)\right]g(x) .$$

Hence, $g(x)|x^n - (1-2v^2)$. The uniqueness of $g(x)$ can be followed from that of $g_1(x), g_2(x)$ and $g_3(x)$. □

In order to study the generator polynomials of the dual codes of $(1-2v^2)$-skew constacyclic codes over R, we give the following definition:

Definition 4.1. *Let $g(x) = g_0 + g_1x + \cdots + g_rx^r$ and $h(x) = h_0 + h_1x + \cdots + h_{n-r}x^{n-r}$ be polynomials in $R[x,\theta_t]$ such that $x^n - (1-2v^2) = h(x)g(x)$ and C' be a $(1-2v^2)$-skew constacyclic code generated by $g(x)$ in $R[x,\theta_t]/\langle x^n - (1-2v^2)\rangle$. Then the dual code of C' is a $(1-2v^2)$-skew constacyclic code generated by the polynomial $\bar{h}(x) = h_{n-r} + \theta_t(h_{n-r-1})x + \cdots + \theta_t^{n-r}(h_0)x^{n-r}$.*

In view of Theorems 3.1 & 4.3, we have the following corollary:

Corollary 4.2. *Let C_1 be a skew cyclic code over F_q and C_2, C_3 be skew negacyclic codes over F_q and $g_1(x)$, $g_2(x)$ and $g_3(x)$ be their generator polynomials such that*

$$x^n - 1 = h_1(x)g_1(x),\ x^n + 1 = h_2(x)g_2(x) = h_3(x)g_3(x) \in F_q[x, \theta_t]\ .$$

If $C = (1 - v^2)C_1 \oplus \frac{p+1}{2}(v^2 + v)C_2 \oplus \frac{p+1}{2}(v^2 - v)C_3$, then

$$C^{\perp} = \langle (1 - v^2)\bar{h_1}(x) + \frac{p+1}{2}(v^2 + v)\bar{h_2}(x) + \frac{p+1}{2}(v^2 - v)\bar{h_3}(x) \rangle$$

and $|C^{\perp}| = q^{deg(g_1(x))+deg(g_2(x))+deg(g_3(x))}$.

5 Idempotent generators of $(1 - 2v^2)$-skew constacyclic codes over R

The idempotent generators of skew cyclic codes over F_q studied by Gursoy et al. [13] under some restrictions. Here we generalise these results for λ-skew constacyclic codes over F_q and the proofs are also similar. So, we are omitting the proof.

Lemma 5.1. *Let $g(x) \in F_q[x, \theta_t]/\langle x^n - \lambda \rangle$ be a monic right divisor of $x^n - \lambda$. If $g.c.d.(n, m_t) = 1$, then $g(x) \in F_{p^t}[x]/\langle x^n - \lambda \rangle$, where $m_t = m/t$ denotes the order of the automorphism θ_t.*

Lemma 5.2. *Let $g(x) \in F_q[x, \theta_t]/\langle x^n - \lambda \rangle$ be a monic right divisor of $x^n - \lambda$ and $C = \langle g(x) \rangle$. If $g.c.d.(n, m_t) = 1$ and $g.c.d.(n, q) = 1$, then there exists an idempotent polynomial $e(x) \in F_q[x, \theta_t]/\langle x^n - \lambda \rangle$ such that $C = \langle e(x) \rangle$.*

Now, we give the idempotent generators of $(1 - 2v^2)$-skew constacyclic codes over R as in the following theorem:

Theorem 5.1. *Let $C = (1 - v^2)C_1 \oplus \frac{p+1}{2}(v^2 + v)C_2 \oplus \frac{p+1}{2}(v^2 - v)C_3$ be a $(1 - 2v^2)$-skew constacyclic code of length n over R and $g.c.d.(n, m_t) = 1$, $g.c.d.(n, q) = 1$. Then C_i has idempotent generator, say $e_i(x)$ for $i = 1, 2, 3$. Moreover*

$$e(x) = (1 - v^2)e_1(x) + \frac{p+1}{2}(v^2 + v)e_2(x) + \frac{p+1}{2}(v^2 - v)e_3(x)$$

is an idempotent generator of C, that is, $C = \langle e(x) \rangle$.

Proof. In the light of Theorem 4.3 and Lemma 5.2, the proof follows. □

Now, we close our discussion with the following examples:

Example 1. *Let $R = F_9 + vF_9 + v^2F_9$ be the ring with $v^3 = v$ and θ be the Frobenius automorphism over F_9, that is, $\theta(r) = r^3$ for any $r \in F_9$, where $F_9 = F_3[\alpha]$, $\alpha^2 + 1 = 0$. Then*

$$x^4 - 1 = (x^2 - 1)(x + \alpha)(x + 2\alpha) \in F_9[x, \theta]\ ,$$

$$x^4 + 1 = (2 + x + x^2)(2 + 2x + x^2) \in F_9[x, \theta] .$$

If $g_1(x) = x^2-1,\ g_2(x) = g_3(x) = (2+x+x^2)$, *then* $C_1 = \langle g_1(x)\rangle$ *is a skew cyclic code over* F_9 *with parameters* $[4, 2, 2]$ *and* $C_2 = \langle g_2(x)\rangle,\ C_3 = \langle g_3(x)\rangle$ *are the skew negacyclic codes over* F_9 *of length* 4. *Therefore, the code* $C = \langle (1 - v^2)g_1(x) + \frac{p+1}{2}(v^2 + v)g_2(x) + \frac{p+1}{2}(v^2 - v)g_3(x)\rangle$ *is a* $(1 - 2v^2)$*-skew constacyclic code of length* 4 *over* R. *Further, the Gray image* $\phi(C)$ *of* C *is a skew cyclic code over* F_9 *of length* 8.

Example 2. *Let* $R = F_9 + vF_9 + v^2F_9$ *be the ring with* $v^3 = v$ *and* θ *be the Frobenius automorphism over* F_9, *that is,* $\theta(r) = r^3$ *for any* $r \in F_9$, *where* $F_9 = F_3[\alpha],\ \alpha^2+\alpha+2 = 0$. *Then*

$$\begin{aligned} x^6 - 1 &= (2 + (2 + \alpha)x + (1 + 2\alpha)x^3 + x^4)(1 + (2 + \alpha)x + x^2) \\ &= (2 + x + (2 + 2\alpha)x^2 + x^3)(1 + x + 2\alpha x^2 + x^3)\ , \\ x^6 + 1 &= (1 + x^2)^3 \in F_9[x, \theta] . \end{aligned}$$

If $g_1(x) = 2+(2+\alpha)x+(1+2\alpha)x^3+x^4$ *and* $g_2(x) = g_3(x) = (1+x^2)^3$, *then* $C_1 = \langle g_1(x)\rangle$ *is a skew cyclic code of length* 6 *over* F_9, $C_2 = \langle g_2(x)\rangle,\ C_3 = \langle g_3(x)\rangle$ *are the skew negacyclic codes of length* 6 *over* F_9. *Thus the code*

$$C = \langle (1 - v^2)g_1(x) + \frac{p+1}{2}(v^2 + v)g_2(x) + \frac{p+1}{2}(v^2 - v)g_3(x)\rangle$$

is a $(1 - 2v^2)$*-skew constacyclic code of length* 6 *over* R. *Also, the Gray image* $\phi(C)$ *of* C *is a skew cyclic code over* F_9 *of length* 12.

Example 3. *Let* $R = F_9 + vF_9 + v^2F_9$ *be the ring with* $v^3 = v$ *and* θ *be the Frobenius automorphism over* F_9, *that is,* $\theta(r) = r^3$ *for any* $r \in F_9$, *where* $F_9 = F_3[\alpha],\ \alpha^2 + 1 = 0$. *Then*

$$x^9 - 1 = (x + 2)^9,\ x^9 + 1 = (x + 1)^9 \in F_9[x, \theta] .$$

Let $g_1(x) = x + 2,\ g_2(x) = g_3(x) = x + 1$. *Then* $C_1 = \langle g_1(x)\rangle$ *is a skew cyclic code of length* 9 *over* F_9 *and* $C_2 = \langle g_2(x)\rangle,\ C_3 = \langle g_3(x)\rangle$ *are the skew negacyclic codes of length* 9 *over* F_9. *Therefore, the code* $C = \langle (1 - v^2)g_1(x) + \frac{p+1}{2}(v^2 + v)g_2(x) + \frac{p+1}{2}(v^2 - v)g_3(x)\rangle$ *is a* $(1 - 2v^2)$*-skew constacyclic code of length* 9 *over* R. *Also, the Gray image* $\phi(C)$ *of* C *is a skew cyclic code of length* 18 *over* F_9.

6 Conclusion

In this paper, we have studied the structural properties of $(1 - 2v^2)$-skew constacyclic codes over the ring $F_q + vF_q + v^2F_q$ by taking the automorphism $\theta_t\colon a + vb + v^2c \mapsto a^{p^t} + vb^{p^t} + v^2c^{p^t}$. We have proved that the Gray image of a $(1 - 2v^2)$-skew constacyclic code of length n over $F_q + vF_q + v^2F_q$ is a skew cyclic code of length $2n$ over F_q. Further, we have obtained idempotent generators of $(1 - 2v^2)$-skew constacyclic codes over $F_q + vF_q + v^2F_q$.

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Generalized total graphs of commutative rings: A survey

Abstract: The study on graphs from commutative rings has become an active field of research for the past three decades. Most popular constructions are zero-divisor graphs and total graphs from commutative rings. Through these constructions, the interplay between algebraic structures and graphs are studied. Several generalizations of total graphs from commutative rings have been studied in the recent past. In this paper, we present a survey on generalized total graphs from commutative rings.

Keywords: Commutative ring; total graph; generalized total graph; prime ideal.

1 Introduction

Algebraic combinatorics is an area of mathematics which employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and has attracted considerable attention. Over the past several years, there has been considerable attention in the literature to associate graphs with commutative rings. There are so many ways to construct graphs from ring structures. Most popular constructions are zero-divisor graphs and total graphs from commutative rings. Through these constructions, the interplay between algebraic structures and graphs are studied. Indeed, it is worthwhile to relate algebraic properties of the commutative rings to the combinatorial properties of the assigned graphs. Note that in the case of zero-divisor graphs, multiplication of the ring is used for adjacency(edges) in the graph. In variation to this, the addition of the ring is used to construct edges in the total graph. The remainder of this survey is devoted to this notion of total graphs. The total graph introduced by Anderson and Badawi [6]) has been investigated in [3, 5, 9, 14, 16, 18, 20]; and several variants of the total graph have been studied in [2, 7, 8, 11, 12, 21, 22, 23, 24, 25]. One can find a detailed survey on total graphs in [10, 17].

Through out this paper, R is a commutative ring with identity, $Z(R)$ denotes the set of all zero-divisors in R, $Z(R)^* = Z(R) \setminus \{0\}$, $U(R)$ is the multiplicative group of units in R and $Reg(R) = R \setminus Z(R)$. The *total graph* of R, denoted by $T(\Gamma(R))$, is the undirected simple graph with all elements of R as vertices, and for two distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. Let $Reg(\Gamma(R))$ be the (induced)

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subgraph of $T(\Gamma(R))$ with vertices $Reg(R)$, which is called *regular graph of a ring* and let $Z(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $Z(R)$. Note that if A is a subring of a commutative ring B, then $T(\Gamma(A))$ need not be an induced subgraph of $T(\Gamma(B))$. Although $x, y \in A$ are adjacent in $T(\Gamma(B))$ if they are adjacent in $T(\Gamma(A))$ since $Z(A) \subseteq Z(B)$, they may be adjacent in $T(\Gamma(B))$, but not adjacent in $T(\Gamma(A))$. In fact, $T(\Gamma(A))$ is an induced subgraph of $T(\Gamma(B))$ if and only if $Z(B) \cap A = Z(A)$.

2 Brief about total graphs

In this section, we briefly mention about the total graph of commutative rings. The study of total graphs breaks naturally into two cases depending on whether or not $Z(R)$ is an ideal of R. First, we sketch the case when $Z(R)$ is an ideal of R (i.e., when $Z(R)$ is closed under addition). Note that since $Z(R)$ is a union of prime ideals of R, we always have $xy \in Z(R)$ for $x, y \in R \Rightarrow x \in Z(R)$ or $y \in Z(R)$. So if $Z(R)$ is an ideal of R, then $Z(R)$ is actually a prime ideal of R, and hence $\frac{R}{Z(R)}$ is an integral domain. Moreover, if R is a finite commutative ring and $Z(R)$ is an ideal of R, then R is local with $Z(R) = Nil(R)$ its unique maximal ideal. Note that if $Z(R)$ is an ideal of R, then $Z(\Gamma(R))$ is a complete subgraph of $T(\Gamma(R))$ and is disjoint from $Reg(\Gamma(R))$. Thus the subgraph $Reg(\Gamma(R))$ was studied instead of $T(\Gamma(R))$. Recall that $Reg(\Gamma(R))$ is the induced subgraph of $T(\Gamma(R))$ with vertices $Reg(R)$. The following is the structure theorem for the total graph of commutative rings.

Theorem 2.1 ([6, Theorem 2.2]). *Let R be a commutative ring such that $Z(R)$ is an ideal of R, and let $|Z(R)| = \alpha$ and $|\frac{R}{Z(R)}| = \beta$.*
(1) *If $2 \in Z(R)$, then $Reg(\Gamma(R))$ is the union of $\beta - 1$ disjoint $K'_{\alpha}s$;*
(2) *If $2 \notin Z(R)$, then $Reg(\Gamma(R))$ is the union of $\frac{\beta-1}{2}$ disjoint $K'_{\alpha,\alpha}s$.*

Next we concentrate on the remaining case; that is $Z(R)$ is not an ideal of R. Since $Z(R)$ is always closed under multiplication by elements of R, this just means that there are distinct $x, y \in Z(R)^*$ such that $x+y \in Reg(R)$. In this case, $Z(\Gamma(R))$ is always connected (but never complete), $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ are never disjoint subgraphs of $T(\Gamma(R))$, and $|Z(R)| \geq 3$. It is shown that $T(\Gamma(R))$ is connected when $Reg(\Gamma(R))$ is connected. However, an ex was given to show that the converse fails.

Theorem 2.2 ([6, Theorem 3.1]). *Let R be a commutative ring such that $Z(R)$ is not an ideal of R.*
(1) *$Z(\Gamma(R))$ is connected with $diam(Z(\Gamma(R))) = 2$;*
(2) *Some vertex of $Z(\Gamma(R))$ is adjacent to a vertex of $Reg(\Gamma(R))$. In particular, the subgraphs $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ of $T(\Gamma(R))$ are not disjoint;*
(3) *If $Reg(\Gamma(R))$ is connected, then $T(\Gamma(R))$ is connected.*

Example 1. Let $R = \mathbb{Q}[x](+)\frac{\mathbb{Q}(x)}{\mathbb{Q}[x]}$. Then one can show that $Z(R) = (\mathbb{Q}[x] \setminus \mathbb{Q}^*)(+)\frac{\mathbb{Q}(x)}{\mathbb{Q}[x]}$ is not an ideal of R and $Reg(R) = U(R) = \mathbb{Q}^*(+)\frac{\mathbb{Q}(x)}{\mathbb{Q}[x]}$. Thus $T(\Gamma(R))$ is connected with $\mathrm{diam}(T(\Gamma(R))) = 2$ (by Theorem 2.3 below) since $R = ((X, 0), (X + 1, 0))$ with $(X, 0), (X + 1, 0) \in Z(R)$. However, $Reg(\Gamma(R))$ is not connected since there is no path from $(1, 0)$ to $(2, 0)$ in $Reg(\Gamma(R))$. It is already observed that $Z(\Gamma(R))$ is connected with $\mathrm{diam}(Z(\Gamma(R))) = 2$.

The next result determines when $T(\Gamma(R))$ is connected and $\mathrm{diam}(T(\Gamma(R)))$ is computed. In particular, $T(\Gamma(R))$ is connected if and only if $\mathrm{diam}(T(\Gamma(R))) < \infty$.

Theorem 2.3 ([6, Theorems 3.3, 3.4]). *Let R be a commutative ring such that $Z(R)$ is not an ideal of R. Then $T(\Gamma(R))$ is connected if and only if $1 = z_1 + \cdots + z_n$ for some $z_1, \ldots, z_n \in Z(R)$. Furthermore, suppose that $T(\Gamma(R))$ is connected and let n be the least integer such that $1 = z_1 + \cdots + z_n$ for some $z_1, \ldots, z_n \in Z(R)$. Then $diam(T(\Gamma(R))) = n$. In particular, if R is a finite commutative ring and $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected and $diam(T(\Gamma(R))) = 2$.*

The total graphs of polynomial rings and idealizations are studied by Pucanović et al. in [18]. Some graphical properties like Euler, Hamiltonian, coloring, connectivity and domination number of the total graph of a commutative ring are discussed by several authors. Eulerian and Hamiltonian nature of the total graphs are studied by Akbari et al. [5], Asir et al. [9] and Shekarriz et al. [20]. The chromatic number and the clique number of the total graph of a commutative ring are studied by Akbari et al.[4] and Aalipour et al. [1]. The clique number of complement of total graphs were studied by Maimani et al. in [13]. The connectivity of the total graph of a commutative ring was studied by Akbari et al. [5] and Ramin et al.[19]. Shekarriz et al. [20] and Tamizh Chelvam et al. [21] have independently studied the domination number of the total graph of a commutative ring. In [8, 25], Asir and Tamizh Chelvam introduced the concept of intersection graph of dominating sets in total graphs and studied several of its properties. Further researchers concentrated on topological properties (like genus) of the total graph of a commutative ring. Miamani et al. [16] initiated the study on the genus of the total graph and they characterized all commutative rings whose total graph has genus either zero or one. Subsequently, Tamizh Chelvam et al. [24] provided some bounds for genus of the total graph and characterized all commutative rings whose total graph has genus two.

3 Generalized total graph of commutative ring

In this section, we present generalizations of total graphs of commutative rings. There are three types of generalizations available and let us present the definitions of all the three generalizations.

Definition 3.1. (1) *Let I be a proper ideal of a commutative ring R, let $S(I)$ be the set of all elements of R that are not prime to I; i.e., $S(I) = \{a \in R : ra \in I \text{ for some } r \in R \setminus I\}$. The total graph of a commutative ring R with respect to proper ideal I, denoted by $T(\Gamma_I(R))$, is the undirected graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in S(I)$. In the case $I = \{0\}$, the graph $T(\Gamma_I(R))$ is the total graph of R.*

(2) *Let R be a commutative ring and S be a multiplicatively closed subset of R. Define a simple graph, denoted by $\Gamma_S(R)$, with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in S$. If we take $S = Z(R)$, then $\Gamma_S(R) = T(\Gamma(R))$.*

(3) *Let R be a commutative ring and H be a nonempty proper subset of R such that $R \setminus H$ is a saturated multiplicatively closed subset of R. The generalized total graph of R is the simple graph $GT_H(R)$ with all elements of R as the vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in H$. When $H = Z(R)$, we have that $GT_H(R)$ is the total graph of R.*

4 Ideal based generalized total graph

In this section, we are interested in the generalized total graph $T(\Gamma_I(R))$ of a commutative ring R with respect to proper ideal I. For an ideal I of R, $S(I) = \{a \in R : ra \in I \text{ for some } r \in R \setminus I\}$. Let $S(\Gamma_I(R))$ be the induced subgraph of $T(\Gamma_I(R))$ with vertices $S(I)$, and let $\overline{S}(\Gamma_I(R))$ be the induced subgraph $T(\Gamma_I(R))$ with vertices $R \setminus S(I)$.

Note that $S(I)$ is not always closed under addition and so not necessarily an ideal of R. Since $S(I)$ is a union of prime ideals of R containing I, whenever $xy \in S(I)$ for $x, y \in R$, then $x \in S(I)$ or $y \in S(I)$. So, if $S(I)$ is an ideal of R, then it is actually a prime ideal of R; hence the study of $T(\Gamma_I(R))$ breaks naturally into two cases depending on whether or not $S(I)$ is an ideal of R. We first list out the relationship between $T(\Gamma_I(R))$ and $T(\Gamma(\frac{R}{I}))$. It is easily to check that $Z(\frac{R}{I}) = \{a + I : a \in S(I)\}$ and $Reg(\frac{R}{I}) = \{a + I : a \notin S(I)\}$. Thus $Z(\frac{R}{I})$ is an ideal of $\frac{R}{I}$ if and only if $S(I)$ is an ideal of R.

Example 2. Let $R = \mathbb{Z}_8$, $S = \mathbb{Z}_4 \times \mathbb{Z}_2$ and $I = \{0, 2, 4, 6\} \trianglelefteq R$, $J = \{0\} \times \mathbb{Z}_2 \trianglelefteq S$. It is easy to check that $S(I) = I$ and $S(J) = \{(0, 0), (0, 1), (2, 0), (2, 1)\}$. Then $T(\Gamma_I(R))$ and $T(\Gamma_J(S))$ are the union of 2 disjoint K_4's (see Figure 3.1). Now, $T(\Gamma(\frac{R}{I}))$ is a graph with two vertices but $T(\Gamma(\frac{S}{J}))$ is a graph with four vertices.

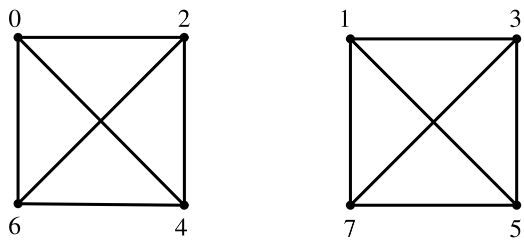

Figure 3.1: $T(\Gamma_{2\mathbb{Z}_8}(\mathbb{Z}_8))$

The theorem given below talks about the adjacency in $T(\Gamma(\frac{R}{I}))$ and $T(\Gamma_I(R))$.

Theorem 4.1 ([2, Theorem 2.3]). *Let R be a commutative ring with the proper ideal I, and let $x, y \in R$. Then*

(1) *If $x + I$ and $y + I$ are (distinct) adjacent vertices in $T(\Gamma(\frac{R}{I}))$, then x is adjacent to y in $T(\Gamma_I(R))$;*
(2) *If x and y are (distinct) adjacent vertices in $T(\Gamma_I(R))$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $T(\Gamma(\frac{R}{I}))$;*
(3) *If x is adjacent to y in $T(\Gamma_I(R))$ and $x + I = y + I$, then $2x, 2y \in S(I)$ and all distinct elements of $x + I$ are adjacent in $T(\Gamma_I(R))$.*

Corollary 4.1 ([2, Corollary 2.4]). *Let R be a commutative ring with the proper ideal I. Then $T(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $T(\Gamma(\frac{R}{I}))$.*

In view of the above corollary, there is a strong relationship between $T(\Gamma(\frac{R}{I}))$ and $T(\Gamma_I(R))$.

Lemma 4.1 ([2, Lemma 2.8]). *Let R be a commutative ring with the proper ideal I. Then $gr(T(\Gamma_I(R))) \leq gr(T(\Gamma(\frac{R}{I})))$. If $T(\Gamma(\frac{R}{I}))$ contains a cycle, then so does $T(\Gamma_I(R))$, and therefore $gr(T(\Gamma_I(R))) \leq gr(T(\Gamma(\frac{R}{I}))) \leq 4$.*

Now we list out the relationship between $T(\Gamma_I(R))$ and $T(\Gamma(\frac{R}{I}))$ with assumption that, $S(I)$ be an ideal of R. If $S(I)$ is an ideal of R, then, by definition, $S(\Gamma_I(R))$ is a complete subgraph $T(\Gamma_I(R))$ and is disjoint from $\overline{S}(\Gamma_I(R))$.

Theorem 4.2 ([2, Theorem 3.3]). *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R.*

(1) *Assume that Γ is an induced subgraph of $\overline{S}(\Gamma_I(R))$ and let x and y be distinct vertices of Γ such that are connected by a path in Γ. Then there exists a path in Γ of length 2 between x and y. In particular, if $\overline{S}(\Gamma_I(R))$ is connected, then $diam(\overline{S}(\Gamma_I(R))) \leq 2$;*
(2) *Suppose x and y are distinct elements of $\overline{S}(\Gamma_I(R))$ that are connected by a path. If $x+y = 2 \notin S(I)$ (that is, if x and y are not adjacent), then $x-(-x)-y$ and $x-(-y)-y$ are paths of length 2 between x and y in $\overline{S}(\Gamma_I(R))$.*

Theorem 4.3 ([2, Theorem 3.4]). *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then the following statements are equivalent.*

(1) $\overline{S}(\Gamma_I(R))$ *is connected;*
(2) *Either* $x + y \in S(I)$ *or* $x - y \in S(I)$ *for all* $x, y \in R \setminus S(I)$*;*
(3) *Either* $x + y \in S(I)$ *or* $x + 2y \in S(I)$ *(but not both) for all* $x, y \in R \setminus S(I)$*. In particular, either* $2x \in S(I)$ *or* $3x \in S(I)$ *for all* $x \in R \setminus S(I)$*.*

Theorem 4.4 ([2, Theorem 3.5]). *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R, and let $|S(I)| = \alpha$ and $|\frac{R}{S(I)}| = \beta$ (we allow α and β to be infinite, then we have $\beta - 1 = (\beta - 1)/2 = \beta$).*
(1) *If $2 \in S(I)$, then $\overline{S}(\Gamma_I(R))$ is the union of $\beta - 1$ disjoint K'_αs;*
(2) *If $2 \notin S(I)$, then $\overline{S}(\Gamma_I(R))$ is the union of $(\beta - 1)/2$ disjoint $K'_{\alpha,\alpha}$s.*

Note that if $S(I) = \{0\}$, then R is an integral domain, and $2 \in S(I)$ if and only if $char(R) = 2$.

Theorem 4.5 ([2, Theorem 3.9]). *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then*
(1) *$\overline{S}(\Gamma_I(R))$ is complete if and only if $\frac{R}{S(I)} \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$;*
(2) *$\overline{S}(\Gamma_I(R))$ is connected if and only if $\frac{R}{S(I)} \cong \mathbb{Z}_2$ or $\frac{R}{S(I)} \cong \mathbb{Z}_3$;*
(3) *$\overline{S}(\Gamma_I(R))$ (and hence $T(\Gamma_I(R))$ and $S(\Gamma_I(R))$) is totally disconnected if and only if $I = \{0\}$ and R is an integral domain, with $char(R) = 2$.*

Next few results give explicit information about the diameter and the girth of $\overline{S}(\Gamma_I(R))$.

Theorem 4.6 ([2, Proposition 3.11]). *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then $diam(\overline{S}(\Gamma_I(R))) \in \{0, 1, 2, \infty\}$ and $gr(\overline{S}(\Gamma_I(R))) \in \{3, 4, \infty\}$.*

Theorem 4.7 ([2, Theorem 3.12]). *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R.*
(1) *$diam(\overline{S}(\Gamma_I(R))) = 0$ if and only if $R \cong \mathbb{Z}_2$;*
(2) *$diam(\overline{S}(\Gamma_I(R))) = 1$ if and only if either $R/S(I) \cong \mathbb{Z}_2$ and $|S(I)| \geq 2$ or $R \cong \mathbb{Z}_2$;*
(3) *$diam(\overline{S}(\Gamma_I(R))) = 2$ if and only if $\frac{R}{S(I)} \cong \mathbb{Z}_3$ and $|S(I)| \geq 2$;*
(4) *Otherwise, $diam(\overline{S}(\Gamma_I(R))) = \infty$.*

Theorem 4.8 ([2, Theorem 3.15]). *Suppose that $S(I)$ is an ideal of R. Then*
(1) (a) *$gr(\overline{S}(\Gamma_I(R))) = 3$ if and only if $2 \in S(I)$ and $|S(I)| \geq 3$;*
(b) *$gr(\overline{S}(\Gamma_I(R))) = 4$ if and only if $2 \notin S(I)$ and $|S(I)| \geq 2$;*
(c) *Otherwise, $gr(\overline{S}(\Gamma_I(R))) = \infty$.*
(2) (a) *$gr(T(\Gamma_I(R))) = 3$ if and only if $|S(I)| \geq 3$;*
(b) *$gr(T(\Gamma_I(R))) = 4$ if and only if $2 \notin S(I)$ and $|S(I)| = 2$;*
(c) *Otherwise, $gr(T(\Gamma_I(R))) = \infty$.*

Now we list the results concerning the remaining case when $S(I)$ is not an ideal of R. Since $S(I)$ is always closed under product by elements of R; hence there are distinct $x, y \in S(I)^*$ such that $x + y \in R \setminus S(I)$, so $|S(I)| \geq 3$; in this case, $S(\Gamma_I(R))$ and $\overline{S}(\Gamma_I(R))$

are never disjoint subgraphs. Also $diam(T(\Gamma_I(R)))$ was computed when $T(\Gamma_I(R))$ is connected.

Theorem 4.9 ([2, Theorem 4.1]). *Suppose that $S(I)$ is not an ideal of R.*
(1) *$S(\Gamma_I(R))$ is connected with $diam(S(\Gamma_I(R))) = 2$;*
(2) *Some vertex of $S(\Gamma_I(R))$ is adjacent to a vertex of $\bar{S}(\Gamma_I(S))$; In particular, the subgraphs $S(\Gamma_I(R))$ and $\overline{S}(\Gamma_I(S))$ are not disjoint.*
(3) *If $\overline{S}(\Gamma_I(S))$ is connected, then $T(\Gamma_I(S))$ is connected.*

Lemma 4.2 ([2, Lemma 4.2]). *Suppose that $S(I)$ is not an ideal of R. Then $T(\Gamma_I(R))$ is connected if and only if $R = \langle a_1, \dots, a_k\rangle$ for some $a_1, \dots, a_k \in S(I)$. In particular, if $\frac{R}{I}$ is a finite ring and $I \subseteq J(R)$, then $T(\Gamma_I(R))$ is connected where $J(R)$ denotes Jacobson radical of R.*

Theorem 4.10 ([2, Theorem 4.3]). *Suppose that $S(I)$ is not an ideal of R and $R = \langle S(I)\rangle$. Let $n \geq 2$ be the least integer such that $R = \langle x_1, \dots, x_n\rangle$ for some $x_1, \dots, x_n \in S(I)$ (that is, $T(\Gamma_I(R))$ is connected). Then $diam(T(\Gamma_I(R))) = n$. In particular, if $\frac{R}{I}$ is a finite ring and $I \subseteq J(R)$, then $diam(T(\Gamma_I(R))) = 2$.*

Clearly, if $R = \langle a_1, \dots, a_k\rangle$ for some $a_1, \dots, a_k \in S(I)$, then $\frac{R}{I} = \langle a_1 + I, \dots, a_k + I\rangle$ and hence $diam(T(\Gamma(\frac{R}{I}))) \leq diam(T(\Gamma_I(R)))$. If $k \geq 2$ is the least integer such that $R = \langle a_1, \dots, a_k\rangle$, then $diam(T(\Gamma(\frac{R}{I}))) \geq diam(T(\Gamma_I(R))) - 1$.

Theorem 4.11 ([2, Theorem 4.5]). *Suppose that $S(I)$ is not an ideal of R. If $T(\Gamma_I(R))$ is connected, then*
(1) *$diam(T(\Gamma_I(R))) = d(0, 1)$;*
(2) *If $diam(T(\Gamma_I(R))) = n$, then $diam(\overline{S}(\Gamma_I(R))) \geq n - 2$.*

Corollary 4.2 ([2, Corollary 4.6]). *Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of commutative rings with $|\Lambda| \geq 2$, and let $R = \prod_{\alpha\in\Lambda} R_\alpha$. Suppose $I = \prod_{\alpha\in\Lambda} I_\alpha$ such that for every $\alpha \in \Lambda$, I_α is a proper ideal of R_α. Then $T(\Gamma_I(R))$ is connected with $diam(T(\Gamma_I(R))) = 2$.*

Theorem 4.12 ([2, Theorem 4.8]). *Let $S(I)$ does not an ideal of R and $S = R \setminus S(I)$. Then $T(\Gamma_{S^{-1}I}(S^{-1}R))$ is connected with $diam(T(\Gamma_{S^{-1}I}(S^{-1}R))) = 2$. In particular, if $\frac{R}{I}$ is a finite ring and $I \subseteq J(R)$, then $diam(T(\Gamma_{S^{-1}I}(S^{-1}R))) = 2$.*

Theorem 4.13 ([2, Theorem 4.9]). *Let $I \trianglelefteq R$, and P_1 and P_2 be prime ideals of R, containing I. Suppose $xy \in I$ for some $x \in P_1\backslash P_2$ and $y \in P_2\backslash P_1$. Then $diam(T(\Gamma_{S^{-1}I}(R_S))) = 2$ where $S = R \setminus P_1 \cup P_2$.*

The following theorem gives the girth for $S(\Gamma_I(R))$, $\overline{S}(\Gamma_I(R))$ and $T(\Gamma_I(R))$ when $S(I)$ is not an ideal of R.

Theorem 4.14 ([2, Theorem 4.10]). *Let R be a commutative ring with the proper ideal I such that $S(I)$ is not an ideal of R. Then*
(1) *If $I \neq 0$, $gr(S(\Gamma_I(R))) = 3$. Otherwise $gr(S(\Gamma_I(R))) = 3$ or ∞. Moreover, if $gr(S(\Gamma_I(R))) = \infty$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$; so, $S(\Gamma_I(R))$ is a $K_{1,2}$ star graph with center 0.*

(2) $gr(T(\Gamma_I(R))) = 3$ *if and only if* $gr(S(\Gamma_I(R))) = 3$*;*
(3) *The (induced) subgraph of* $S(\Gamma_I(R))$ *with vertices* $\sqrt{I}$ *is complete; hence* $gr(S(\Gamma_I(R))) = 3$ *when* $|\sqrt{I}| \geq 3$*;*
(4) *If* $gr(T(\Gamma_I(R))) = 4$*, then* $gr(S(\Gamma_I(R))) = \infty$*;*
(5) *If* $2 \in I$*, then* $gr(\overline{S}(\Gamma_I(R))) = 3$ *or* ∞*;*
(6) *If* $2 \notin I$*, then* $gr(\overline{S}(\Gamma_I(R))) = 3, 4$ *or* ∞*.*

5 Generalized total graph from multiplicatively closed sets

Next we present the second generalization of the total graph of a commutative ring. Let R be a commutative ring. A subset S of R which is closed under multiplication is called *multiplicatively closed*. In 2012, Z. Barati et al. [11] introduced the graph $\Gamma_S(R)$ associated to a commutative ring R and a multiplicatively closed subset S of R. The graph $\Gamma_S(R)$ is a simple graph with all elements of R as vertices, and two distinct vertices x and y of R are adjacent if and only if $x + y \in S$. Since the subsets $Z(R)$ and $U(R)$ of R are multiplicatively closed, $\Gamma_S(R)$ is a natural generalization of the total graph and the unit graph of R.

First we observe certain relationships of the associated graphs $\Gamma_S(R)$ with the total graph, unit graph, and some Cayley graphs. Recall that the *unit graph* of R, defined as the simple graph with all elements of R as vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in U(R)$. Let H be a finite group with identity element e and let T be a subset of H such that $e \notin T$ and $T^{-1} = \{x^{-1} : x \in T\} \subseteq T$. Then the *Cayley graph* associated to H and T, denoted by $Cay(H, T)$, is a simple graph with all elements of H as vertices, and two distinct vertices x and y of H are adjacent if and only if $xy^{-1} \in T$. In the following example we show that, for a positive integer n, $Cay(\mathbb{Z}_{2n}, \{1, -1\}) \cong \Gamma_{\{1,-1\})}(\mathbb{Z}_{2n})$.

Example 3. *Let* $R = \mathbb{Z}_{2n}$ *and* $S = \{1, -1\} \subseteq R$*. Then* S *is a multiplicatively closed subset of* R*. Considering* $(\mathbb{Z}_{2n}, +)$ *as a group, so we can define* $Cay(\mathbb{Z}_{2n}, \{1, -1\})$*. We show that two graphs* $\Gamma_{\{1,-1\}}(\mathbb{Z}_{2n})$ *and* $Cay(\mathbb{Z}_{2n}, \{1, -1\})$ *are isomorphic. For, consider the map* $f : \Gamma_{\{1,-1\}}(\mathbb{Z}_{2n}) \to Cay(\mathbb{Z}_{2n}, \{1, -1\})$ *given by* $f(x) = x$*, if* x *is even and* $f(x) = -x$*, otherwise. Clearly* f *is a bijection. Now, we show that* f *is a homomorphism. To achieve this, suppose that* $\{x, y\}$ *is an edge in* $\Gamma_{\{1,-1\}}(\mathbb{Z}_{2n})$*. So both* x *and* y *are neither even nor odd. Hence without loss of generality, we may assume that* x *is even and* y *is odd and that the sum of* x *and* y *are* 1 *or* -1*. Therefore* $x - (-y)$ *is equal to* 1 *or* -1*. This means that* $f(x)$ *and* $f(y)$ *are adjacent in* $Cay(\mathbb{Z}_{2n}, \{1, -1\})$*. Conversely, if* $\{x, y\}$ *is an edge in* $Cay(\mathbb{Z}_{2n}, \{1, -1\})$*, then* $x - y$ *is equal to* 1 *or* -1*. Hence without loss of generality, we may assume that* x *is odd and* y *is even. Since* $f(-x) = x$ *and* $f(y) = y$*, we have that* $-x - y$

is equal to 1 *or* -1*, which implies that the corresponding vertices* x *and* y *are adjacent in* $\Gamma_{\{1,-1\}}(\mathbb{Z}_{2n})$.

Now we present some basic properties of $\Gamma_S(R)$. Let us start with the degree of the vertices in $\Gamma_S(R)$.

Lemma 5.1 ([11, Lemma 1.3]). *Suppose that* S *is an arbitrary multiplicatively closed subset of* R*. Then, in the graph* $\Gamma_S(R)$,
(1) *for each* $x \in R$ *with* $x + x \notin S$*, we have* $deg(x) = |S|$*, and*
(2) *for each* $x \in R$ *with* $x + x \in S$*, we have* $deg(x) = |S| - 1$*. In particular, if* $2 \in S$*, then* $deg(x) = |S| - 1$ *for all* $x \in S$.

Next we present a characterization for the graph $\Gamma_S(R)$ with the assumption that S is an ideal of R.

Theorem 5.1 ([11, Theorem 1.4]). Suppose that S is an ideal of R with $|S| = \alpha$. Set $A = \{x + S\colon x \in R \setminus S \text{ and } 2x \in S\}$ and $B = \{x + S\colon x \in R \setminus S \text{ and } 2x \notin S\}$. Then $\Gamma_S(R)$ is the disjoint union of $|A| + 1$ times K_α and $|B|/2$ times $K_{\alpha,\alpha}$.

As a consequence of Theorem 5.1, we have the following corollary which is an improved form of Theorem 2.1.

Corollary 5.1 ([11, Corollary 1.6]). *Suppose that* S *is an ideal of* R *with* $|S| = \alpha$ *and* $|R/S| = \beta$.
(1) *If* $2 \in S$*, then* $\Gamma_S(R)$ *is the union of* β *disjoint* K_α*'s;*
(2) *If* $2x \notin S$ *for each* $x \in R \setminus S$*, then* $\Gamma_S(R)$ *is the union of* K_α *with* $(\beta - 1)/2$ *disjoint* $K_{\alpha,\alpha}$*'s.*

Now, we see a necessary and sufficient condition for the connectedness of the graph $\Gamma_S(R)$ in the case that $S = -S$. Since $Z(R)$ and $U(R)$ fulfill this condition, the following theorem can be considered as an improved form of Theorem 2.3 and [15, Proposition 3.2].

Theorem 5.2 ([11, Theorem 1.7]). *Let* S *be a multiplicatively closed subset of* R *such that* $S = -S$*. Then* $\Gamma_S(R)$ *is connected if and only if* $(R, +)$ *is generated by* S.

Note that S in Theorem 5.1 is not necessarily an ideal of R. For instance, if $R = \mathbb{Z}$ and $S = \{1, -1\}$, then $\Gamma_S(R)$ is connected.

Corollary 5.2 ([11, Corollary 1.8]). *For a proper ideal* S *of* R*, the graph* $\Gamma_S(R)$ *is disconnected.*

In the case that both subsets S and $S^c = R \setminus S$ of R are multiplicatively closed, we have the following description for the graphs associated to them. In this case, it is worthwhile to study the relationship between the associated graphs and the multiplicatively closed subsets S and S^c.

Theorem 5.3 ([11, Theorem 1.9]). *Suppose that S and $S^c = R \setminus S$ are two multiplicatively closed subsets of R. Then the complement of $\Gamma_S(R)$ is isomorphic to $\Gamma_{S^c}(R)$.*

Since R is finite, it is the disjoint union of $Z(R)$ and $U(R)$. Thus, as a consequence of Theorem 5.2, we observe the relationship between the unit graph and the total graph of a finite ring.

Corollary 5.3 ([11, Corollary 1.10]). *The complement of the unit graph of a finite ring R is isomorphic to its total graph.*

Suppose that M is an R-module. As mentioned earlier, the idealization $R(+)M$ of M over R is a commutative ring. If S is a multiplicatively closed subset of R, then it is easy to see that $\hat{S} := S(+)M$ is a multiplicatively closed subset of $R(+)M$. The following theorem compares the diameter of the graph $\Gamma_{\hat{S}}(R(+)M)$ with the diameter of $\Gamma_S(R)$.

Theorem 5.4 ([11, Theorem 1.11]). *For an $R-module$ M, $\Gamma_{\hat{S}}(R(+)M)$ is connected if and only if $\Gamma_S(R)$ is connected. More precisely, $diam(\Gamma_{\hat{S}}(R(+)M)) = diam(\Gamma_S(R))$.*

Next we consider the case that S is a saturated multiplicatively closed subset of R. A multiplicatively closed subset S of R is called saturated if $xy \in S$ implies that $x \in S$ and $y \in S$. Note that the set of all unit elements $U(R)$ of R is a saturated multiplicatively closed subset of R. On the other hand, if S is a saturated multiplicatively closed subset of R, then, for an arbitrary element s in S, $1 \cdot s \in S$. Hence $1 \in S$. Thus, for any $u \in U(R)$, $uu^{-1} = 1 \in S$, which implies that $u \in S$, and so $U(R) \subseteq S$. This means that $U(R)$ is the smallest saturated multiplicatively closed subset of R. Hereafter in this section, S is a saturated multiplicatively closed subset of R, and so our results about the graph $\Gamma_S(R)$ are natural generalizations of the corresponding results for the unit graph. The following results provide a characterization for the completeness of the graphs $\Gamma_S(R)$.

Lemma 5.2 ([11, Proposition 2.1]). *The graph $\Gamma_S(R)$ is complete if and only if $S = R$ or $char(R) = 2$ and $S = R \setminus \{0\}$.*

Note that if $0 \in S$, then $S = R$, and so, by Lemma 5.2, the graph $\Gamma_S(R)$ is complete. Hence in the rest of this section, we will assume that $0 \notin S$.

Lemma 5.3 ([11, Lemma 2.3]). *For an arbitrary saturated multiplicatively closed subset S of R, in the graph $\Gamma_S(R)$, the following statements hold:*
1. *If $x \in R \setminus S$, then $deg(x) = |S|$;*
2. *If $2 \notin S$, then $deg(x) = |S|$ for all $x \in R$.*

Lemma 5.4 ([11, Lemma 2.4]). *Let x and y be two elements of R. Then the following statements are equivalent:*
1. *x is adjacent to y in $\Gamma_S(R)$;*
2. *$x + I$ is adjacent to $y + I$ in $\Gamma_{\bar{S}}(\frac{R}{I})$;*
3. *Each element of $x + I$ is adjacent to each element of $y + I$ in $\Gamma_S(R)$;*
4. *There exist $x + i$ in $x + I$ and $y + i'$ in $y + I$ which are adjacent in $\Gamma_S(R)$.*

Corollary 5.4 ([11, Corollary 2.5]). *The graph $\Gamma_S(R)$ is connected if and only if $\Gamma_{\bar{S}}(\frac{R}{I})$ is connected.*

Theorem 5.5 ([11, Theorem 2.7]). *The following statements hold:*
(1) $gr(\Gamma_S(R)) \leq gr(\Gamma_{\bar{S}}(\frac{R}{I}))$;
(2) $diam(\Gamma_{\bar{S}}(\frac{R}{I})) \leq diam(\Gamma_S(R))$;
(3) *If* $\Gamma_{\bar{S}}(\frac{R}{I})$ *is a complete graph, then* $diam(\Gamma_S(R)) \leq 2$;
(4) *If* $\Gamma_{\bar{S}}(\frac{R}{I})$ *is not a complete graph, then* $diam(\Gamma_{\bar{S}}(\frac{R}{I})) = diam(\Gamma_S(R))$.

The following ex shows that we may have strict inequality in part (i) of Theorem 5.5. In our example, $\Gamma_{\bar{S}}(\frac{R}{I})$ is a complete graph, but $\Gamma_S(\frac{R}{I})$ is not a complete graph by Theorem 5.5(3).

Example 4 ([11, Example 2.8]). *Let $R = \mathbb{Z}_2[x]$ be the polynomial ring in an indeterminate x with coefficients in $\mathbb{Z}_2$, and set $S := R \setminus (x)$. Hence $I = (x)$, and so $|\frac{R}{I}| = 2$. Set $V_1 := \{a_1x + \ldots + a_nx_n : n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{Z}_2\}$ and $V_2 := \{1 + b_1x + \ldots + b_mx_m : m \in \mathbb{N}$ and $b_1, \ldots, b_m \in \mathbb{Z}_2\}$. Clearly the sum of two elements of V_1 (or V_2) is in (x). This implies that the vertices in the same part are not adjacent. Moreover, for each two vertices $f = a_1x + \ldots + a_nx^n \in V_1$ and $g = 1 + b_1x + \ldots + b_mx^m \in V_2$, we have that $f + g \in S$. This means that $\Gamma_S(R)$ is a complete bipartite graph, and so $diam(\Gamma_S(R)) = 2$. On the other hand, $\Gamma_{\bar{S}}(\frac{R}{I})$ is a K_2, and so $diam(\Gamma_{\bar{S}}(\frac{R}{I})) = 1$. Hence the upper bound given in Theorem 5.5(iii) is sharp.*

Moreover, one can see that $gr(\Gamma_{\bar{S}}(\frac{R}{I})) = \infty$ and $gr(\Gamma_S(R)) = 4$. These facts show that the converse of Theorem 5.5(i) does not hold. Hence, in general, the equality in part (i) of Theorem 5.5 does not hold.

Theorem 5.6 ([11, Theorem 2.12]). *For an R-module M, we have the following statements.*
(1) *If* $2 \notin S$, *then* $\Gamma_{\hat{S}}(R(+)M) = \bigoplus_{|M|^2} \Gamma_S(R)$;
(2) *If* $2 \in S$, *then* $\Gamma_{\hat{S}}(R(+)M) = \bigoplus_{|M|^2} \Gamma_S(R) \oplus (\bigoplus_{|S|} K_{|M|})$.

Proposition 5.1 ([11, Proposition 2.13]). *Let S be a saturated multiplicatively closed subset of R with $R \setminus S = \cup_{i=1}^{n} P_i$ such that $|\frac{R}{P_i}| = 2$ for some i. Then $\Gamma_S(R)$ is a bipartite graph. Furthermore, $\Gamma_S(R)$ is a complete bipartite graph if and only if $n = 1$.*

The next theorem gives us the girth value of $\Gamma_S(R)$.

Theorem 5.7 ([11, Theorem 2.15]). *Let R be finite and S be a saturated multiplicatively closed subset of R. Then $gr(\Gamma_S(R)) \in \{3, 4, 6, \infty\}$.*

The next theorem characterizes the rings R with a saturated multiplicatively closed set S such that the graph $G_S(R)$ has infinite girth, or equivalently, $\Gamma_S(R)$ is a forest.

Theorem 5.8 ([11, Theorem 2.17]). *Let R be finite and S be a saturated multiplicatively closed subset of R. Then $gr(\Gamma_S(R)) = \infty$ if and only if one of the following statements holds:*

(1) $R = \mathbb{Z}_3$;
(2) $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ *and* $|S| = 1$.

By Theorem 5.8, one can obtain the following characterization for a ring R with saturated multiplicatively closed set S such that the graph $\Gamma_S(R)$ is forest.

Corollary 5.5 ([11, Corollary 2.18]). *Let R be a finite ring such that $R \neq \mathbb{Z}_3$. Also, suppose that S is a saturated multiplicatively closed subset of R. Then $\Gamma_S(R)$ is forest if and only if it is a complete matching.*

In some special cases, we will find a better upper bound for the girth of the graph $\Gamma_S(R)$. In the following theorem, we study the local case.

Theorem 5.9 ([11, Theorem 2.19]). *Suppose that $(R, \mathfrak{m})$ is a local ring and that $R \setminus S = \bigcup_{i \in A} P_i$. Then*
(1) *If $|A| \geq 2$, then $gr(\Gamma_S(R)) = 3$;*
(2) *If $|A| = 1$, then $gr(\Gamma_S(R)) \leq 4$.*

Lemma 5.5 ([11, Lemma 2.20]). *Let $(R, \mathfrak{m})$ be a finite local ring and S be a saturated multiplicatively closed subset of R. Then $S = U(R)$.*

Let R be a finite ring. We can write $R = R_1 \times R_2 \times \ldots \times R_k$ such that every R_i is a finite local ring with maximal ideal $\mathfrak{m}_i$. Now, let S be a saturated multiplicatively closed subset of R. It is not hard to see that $S = S_1 \times S_2 \times ł \times S_k$, where, for $1 \leq i \leq k$, $S_i = \{s_i \in R_i : (s_1, ł, s_{i-1}, s_i, s_{i+1}, ł, s_k) \in S\}$ is a saturated multiplicatively closed subset of R_i or $S_i = R_i$.

Theorem 5.10 ([11, Proposition 2.21]). *Let $R = R_1 \times R_2 \times \ldots \times R_k$, where $(R_i, \mathfrak{m}_i)$ is a finite local ring such that $\frac{R_i}{\mathfrak{m}_i} \cong \mathbb{Z}_2$, and let $S = S_1 \times S_2 \times \ldots \times S_k$ be a saturated multiplicatively closed subset of R. Then $diam(\Gamma_S(R)) \in \{1, 2, \infty\}$.*

Corollary 5.6. *Let $R = R_1 \times R_2 \times \ldots \times R_k$, where $(R_i, \mathfrak{m}_i)$ is a finite local ring such that $\frac{R_i}{\mathfrak{m}_i} \cong \mathbb{Z}_2$, and let $S = S_1 \times S_2 \times \ldots \times S_k$ be a saturated multiplicatively closed subset of R. Then $\Gamma_S(R)$ is disconnected if and only if there exist $1 \leq i \neq j \leq n$ such that $S_i = U(R_i)$ and $S_j = U(R_j)$.*

Theorem 5.11 ([11, Theorem 2.23]). *Let R be a finite ring. For a saturated multiplicatively closed subset S of R, we have that $diam(\Gamma_S(R)) \in \{1, 2, 3, \infty\}$.*

As an immediate consequence of Theorem 5.11, we have the following characterization for disconnected graphs.

Corollary 5.7. *Let $R = R_1 \times \ldots \times R_n$ be a finite ring and S be a saturated multiplicatively closed subset of R. Then $\Gamma_S(R)$ is disconnected if and only if there exist $i \neq j$ where R_i and R_j have $\mathbb{Z}_2$ as a quotient and $S_i = U(R_i)$ and $S_j = U(R_i)$.*

6 Generalized total graph from multiplicative prime sets

Let R be a commutative ring with nonzero identity. A subset H of R to be a *multiplicative-prime* subset of R if the following two conditions hold: (i) $ab \in H$ for every $a \in H$ and $b \in R$; (ii) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$. For example, H is multiplicative-prime subset of R if H is a prime ideal of R, H is a union of prime ideals of R, $H = Z(R)$, or $H = R \setminus U(R)$. In fact, it is easily seen that H is a multiplicative-prime subset of R if and only if $R \setminus H$ is a saturated multiplicatively closed subset of R. Thus H is a multiplicative-prime subset of R if and only if H is a union of prime ideals of R. Note that if H is a multiplicative-prime subset of R, then $Nil(R) \subseteq H \subseteq R \setminus U(R)$; and if H is also an ideal of R, then H is necessarily a prime ideal of R. In particular, if $R = Z(R) \cup U(R)$ (e.g., R is finite), then $Nil(R) \subseteq H \subseteq Z(R)$.

Let H be a multiplicative-prime subset of a commutative ring R. The *generalized total graph* of R, denoted by $GT_H(R)$, as the (simple) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in H$. For $A \subseteq R$, let $GT_H(A)$ be the induced subgraph of $GT_H(R)$ with all elements of A as the vertices. For example, $GT_H(R \setminus H)$ is the induced subgraph of $GT_H(R)$ with vertices $R \setminus H$. When $H = Z(R)$, we have that $GT_H(R)$ is the so-called total graph. As to be expected, $GT_H(R)$ and $T(\Gamma(R))$ share many properties; However, the concept of generalized total graph, unlike the earlier concept of total graph, allows us to study graphs of integral domains.

Let H be a multiplicative-prime subset of a commutative ring R. Since H is a union of prime ideals of R, the study of $GT_H(R)$ breaks naturally into two cases depending on whether or not H is an (prime) ideal of R. First we handle the case when H is an ideal of R.

The next theorem gives a complete description of $GT_H(R)$ in case of H is an ideal of R. It also shows that non-isomorphic rings may have isomorphic graphs. We allow α and β to be infinite cardinals; if β is infinite, then $\beta - 1 = (\beta - 1)/2 = \beta$.

Theorem 6.1 ([7, Theorem 2.2]). *Let H be a prime ideal of a commutative ring R, and let $|H| = \alpha$ and $|\frac{R}{H}| = \beta$.*
(1) *If $2 \in H$, then $GT_H(R \setminus H)$ is the union of $\beta - 1$ disjoint K'_αs;*
(2) *If $2 \notin H$, then $GT_H(R \setminus H)$ is the union of $(\beta - 1)/2$ disjoint $K'_{\alpha,\alpha}$s.*

From the above theorem, one can easily deduce when $GT_H(R \setminus H)$ is complete or connected, and one can explicitly compute its diameter and girth. First result of this kind determine when $GT_H(R \setminus H)$ is either complete or connected.

Theorem 6.2 ([7, Theorem 2.3]). *Let H be a prime ideal of a commutative ring R.*
(1) *$GT_H(R \setminus H)$ is complete if and only if either $\frac{R}{H} \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$;*
(2) *$GT_H(R \setminus H)$ is connected if and only if either $\frac{R}{H} \cong \mathbb{Z}_2$ or $\frac{R}{H} \cong \mathbb{Z}_3$;*

(3) $GT_H(R \setminus H)$ *(and hence* $GT_H(H)$ *and* $GT_H(R)$*) is totally disconnected if and only if* $H = \{0\}$ *(thus R is an integral domain) and* $char(R) = 2$.

Next result compute both the diameter and girth of $GT_H(R\setminus H)$ when H is a prime ideal of R.

Theorem 6.3 ([7, Theorem 2.4]). *Let H be a prime ideal of a commutative ring R.*
(1) $diam(GT_H(R\setminus H)) = 0, 1, 2,$ *or* ∞. *In particular,* $diam(GT_H(R\setminus H)) \leq 2$ *if* $GT_H(R\setminus H)$ *is connected;.*
(2) $gr(GT_H(R\setminus H)) = 3, 4,$ *or* ∞. *In particular,* $gr(GT_H(R\setminus H)) \leq 4$ *if* $GT_H(R\setminus H)$ *contains a cycle.*

The next theorem gives a more explicit description of the diameter and girth of $GT_H(R\setminus H)$ when H is a prime ideal of R.

Theorem 6.4 ([7, Theorem 2.5]). *Let H be a prime ideal of a commutative ring R.*
(1) (a) $diam(GT_H(R \setminus H)) = 0$ *if and only if* $R \cong \mathbb{Z}_2$;
 (b) $diam(GT_H(R\setminus H)) = 1$ *if and only if either* $R/H \cong \mathbb{Z}_2$ *and* $R \ncong \mathbb{Z}_2$ *(i.e.,* $R/H \cong \mathbb{Z}_2$ *and* $|H| \geq 2$*), or* $R \cong \mathbb{Z}_3$;
 (c) $diam(GT_H(R \setminus H)) = 2$ *if and only if* $R/H \cong \mathbb{Z}_3$ *and* $R \ncong \mathbb{Z}_3$ *(i.e.,* $R/H \cong \mathbb{Z}_3$ *and* $|H| \geq 2$*).*
 (a) Otherwise, $diam(GT_H(R \setminus H)) = \infty$.
(2) (a) $gr(GT_H(R \setminus H)) = 3$ *if and only if* $2 \in H$ *and* $|H| \geq 3$.
 (b) $gr(GT_H(R \setminus H)) = 4$ *if and only if* $2 \notin H$ *and* $|H| \geq 2$;
 (c) *Otherwise,* $gr(GT_H(R \setminus H)) = \infty$.
(3) (a) $gr(GT_H(R)) = 3$ *if and only if* $|H| \geq 3$;
 (b) $gr(GT_H(R)) = 4$ *if and only if* $2 \notin H$ *and* $|H| = 2$;
 (c) *Otherwise,* $gr(GT_H(R)) = \infty$.

The following examples illustrate the previous theorem.

Example 5 ([7, Example 2.6]). *(a) Let* $R = \mathbb{Z}$ *and H be a prime ideal of R. Then* $GT_H(R\setminus H)$ *is complete if and only if* $H = 2\mathbb{Z}$*, and* $GT_H(R \setminus H)$ *is connected if and only if either* $H = 2\mathbb{Z}$ *or* $H = 3\mathbb{Z}$*. Moreover,* $diam(GT_H(R \setminus H)) = 1$ *if and only if* $H = 2\mathbb{Z}$*, and* $diam(GT_H(R \setminus H)) = 2$ *if and only if* $H = 3\mathbb{Z}$*. Let* $p \geq 5$ *be a prime integer and* $H = p\mathbb{Z}$*. Then* $GT_H(R \setminus H)$ *is the the union of* $(p-1)/2$ *disjoint* $K_{\omega,\omega}$*'s; so* $diam(GT_H(R \setminus H)) = \infty$*. Finally,* $diam(GT_H(R \setminus H)) = \infty$ *when* $H = \{0\}$.

Also, $gr(GT_H(R \setminus H)) = \infty$ *if* $H = \{0\}$*,* $gr(GT_H(R \setminus H)) = 3$ *if* $H = 2\mathbb{Z}$*, and* $gr(GT_H(R\setminus H)) = 4$ *otherwise. Moreover,* $gr(GT_{\{0\})}(R)) = \infty$ *and* $gr(GT_H(R)) = 3$ *for any nonzero prime ideal H of R.*

(b) Let $R = \mathbb{Z}_{pm} \times R_1 \times \cdots \times R_n$*, where* $m \geq 2$ *is an integer, p is a positive prime integer, and* $R_1, \ldots, R_n$ *are commutative rings. Then* $H = p\mathbb{Z}_{pm} \times R_1 \times \cdots \times R_n$ *is a prime ideal of R. The graph* $GT_H(R \setminus H)$ *is complete if and only if* $p = 2$*, and* $GT_H(R \setminus H)$ *is connected if and only if* $p = 2$ *or* $p = 3$*. Moreover,* $diam(GT_H(R \setminus H)) = 1$ *if and only if* $p = 2$*, and*

$diam(GT_H(R \setminus H)) = 2$ if and only if $p = 3$. Assume that $p \geq 5$. Then $GT_H(R \setminus H)$ is the union of $(p-1)/2$ disjoint $K_{\alpha,\alpha}$'s, where $\alpha = m|R_1|\cdots|R_n|$; so $diam(GT_H(R \setminus H)) = \infty$.

Also, $gr(GT_H(R \setminus H)) = 3$ if $p = 2$ and $gr(GT_H(R \setminus H)) = 4$ otherwise. Moreover, $gr(GT_H(R)) = 3$ for any prime p.

We have already observed in Theorem 6.1 that $GT_H(H)$ is always connected and $GT_H(R)$ is never connected when H is an ideal of R. The next theorem gives several new criteria for when $GT_H(R \setminus H)$ is connected.

Theorem 6.5 ([7, Theorem 2.8]). *Let H be a prime ideal of a commutative ring R. Then the following statements are equivalent.*
1. *$GT_H(R \setminus H)$ is connected;*
2. *Either $x + y \in H$ or $x - y \in H$ for every $x, y \in R \setminus H$;*
3. *Either $x + y \in H$ or $x + 2y \in H$ for every $x, y \in R \setminus H$. In particular, either $2x \in H$ or $3x \in H$ (but not both) for every $x \in R \setminus H$;*
4. *Either $\frac{R}{H} \cong \mathbb{Z}_2$ or $\frac{R}{H} \cong \mathbb{Z}_3$.*

Now we consider the case when the multiplicative-prime subset H is not an ideal of R. Since H is always closed under multiplication by elements of R, this just means that $0 \in H$ and there are distinct $x, y \in H^*$ such that $x + y \in R \setminus H$. In this case, $GT_H(H)$ is always connected (but never complete), $GT_H(H)$ and $GT_H(R \setminus H)$ are never disjoint subgraphs of $GT_H(R)$, and $|H| \geq 3$.

Theorem 6.6 ([7, Theorem 3.1(3)]). *Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R. If $GT_H(R \setminus H)$ is connected, then $GT_H(R)$ is connected.*

Next, we determine when $GT_H(R)$ is connected and compute $diam(GT_H(R))$. In particular, $GT_H(R)$ is connected if and only if $diam(GT_H(R)) < \infty$.

Theorem 6.7 ([7, Theorem 3.2, Theorem 3.4]). *Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R. Then $GT_H(R)$ is connected if and only if $1 = z_1 + \cdots + z_n$, for some $z_1, \ldots, z_n \in H$. In particular, if H is not an ideal of R and either $dim(R) = 0$ (e.g., R is finite) or R is an integral domain with $dim(R) = 1$, then $GT_H(R)$ is connected. Furthermore, suppose that $G_H(R)$ is connected. Let $n \geq 2$ be the least integer such that $1 = z_1 + \cdots + z_n$ for some $z_1, \ldots, z_n \in H$. Then $diam(GT_H(R)) = n$. In particular, if H is not an ideal of R and either $dim(R) = 0$ (e.g., R is finite) or R is an integral domain with $dim(R) = 1$, then $diam(GT_H(R)) = 2$.*

Theorem 6.8 ([7, Corollary 3.5]). *Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R such that $GT_H(R)$ is connected.*
1. *$diam(GT_H(R)) = d(0, 1)$;*
2. *If $diam(GT_H(R)) = n$, then $diam(GT_H(R \setminus H)) \geq n - 2$.*

The following is an example of a ring R such that $GT_H(R)$ is connected, but $GT_H(R \setminus H)$ is not connected.

Example 6. *(a) Let $R = \mathbb{Q}[X]$. Then $H = \mathbb{Q}[X] \setminus \mathbb{Q}^*$ is a multiplicative-prime subset of R that is not an ideal of R. Thus $GT_H(H)$ is connected with $diam(GT_H(H)) = 2$. Moreover, $GT_H(R)$ is connected with $diam(GT_H(R)) = 2$ (by Theorem 6.6) since $R = (X, X+1)$ with $X, (X+1) \in H$. However, $GT_H(R \setminus H)$ is not connected since there is no path from 1 to 2 in $GT_H(R \setminus H)$. Thus the converse of Theorem 6.6 need not hold.*

(b) Let $R = \mathbb{Z}$. Then $H = \mathbb{Z} \setminus U(\mathbb{Z})$ is a multiplicative-prime subset of R that is not an ideal of R. Since $GT_H(R \setminus H)$ is clearly connected, $GT_H(R)$ is connected by Theorem 6.6.

Theorem 6.9 ([7, Theorem 3.9]). *Let R be a commutative ring and H be a multiplicative-prime subset of R that contains two co-maximal ideals of R. Then $GT_H(R)$ is connected with $diam(GT_H(R)) = 2$.*

The following is an example of a commutative ring R with a multiplicative-prime subset H such that neither $GT_H(R \setminus H)$ nor $GT_H(R)$ is connected, but $GT_{H_S}(R_S)$ is connected for some multiplicatively closed subset S of R with $S \neq R \setminus H$.

Example 7. *Let $R = \mathbb{Z}[X]$, $H = X\mathbb{Z}[X] \cup 3\mathbb{Z}[X]$, and $S = \{1, (X+3), (X+3)^2, (X+3)^3, \ldots\} \subsetneq R \setminus H$. Then H is a multiplicative-prime subset of R that is not an ideal of R and $(H) = (3, X) \subsetneq R$. Thus $GT_H(R)$ is not connected by Theorem 6.7, and hence $GT_H(R \setminus H)$ is not connected by Theorem 6.6. Since $X\mathbb{Z}[X]_S$, $3\mathbb{Z}[X]_S$ are co-maximal ideals of R_S and $X\mathbb{Z}[X]_S, 3\mathbb{Z}[X]_S \subseteq H_S$, the graph $GT_{H_S}(R_S)$ is connected with $diam(GT_{H_S}(R_S)) = 2$ by Theorem 6.8.*

We next investigate the girth of $GT_H(H)$, $GT_H(R \setminus H)$, and $GT_H(R)$ when H is not an ideal of R. Recall that $|H| \geq 3$ if H is not an ideal of R.

Theorem 6.10 ([7, Theorem 3.14]). *Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R.*

1. *Either $gr(GT_H(H)) = 3$ or $gr(GT_H(H)) = \infty$. Moreover, if $gr(GT_H(H)) = \infty$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H = Z(R)$; so $GT_H(H)$ is a $K_{1,2}$ star graph with center 0;*
2. *$gr(GT_H(R)) = 3$ if and only if $gr(GT_H(H)) = 3$;*
3. *$gr(GT_H(R)) = 4$ if and only if $gr(GT_H(H)) = \infty$ (if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$);*
4. *If $char(R) = 2$, then $gr(GT_H(R \setminus H)) = 3$ or ∞. In particular, $gr(GT_H(R \setminus H)) = 3$ if $char(R) = 2$ and $GT_H(R \setminus H)$ contains a cycle;*
5. *$gr(GT_H(R \setminus H)) = 3, 4,$ or ∞. In particular, $gr(GT_H(R \setminus H)) \leq 4$ if $GT_H(R \setminus H)$ contains a cycle.*

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Differential conditions for which near-rings are commutative rings

Abstract: In the present paper, we study the commutativity of near-rings satisfying certain algebraic identities involving two sided α-n-derivations on semigroup ideals.

Keywords: 3-prime near-rings; two sided α-n-derivation; derivations; commutativity.

1 Introduction

In this paper, $\mathcal{N}$ will denote a zero symmetric left near-ring with multiplicative center $Z(\mathcal{N})$. As usual for all $x, y \in \mathcal{N}$, the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$. $\mathcal{N}$ is said to be 2-torsion free, if whenever $2x = 0$ implies $x = 0$. A near-ring $\mathcal{N}$ is called zero symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that the left distributivity yields that $x0 = 0$ for all $x \in \mathcal{N}$). Recall that $\mathcal{N}$ is 3-prime, that is, for all $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. A non empty subset U of $\mathcal{N}$ is said to be a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $U\mathcal{N} \subseteq U$) and if U is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. An additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is said to be a derivation on $\mathcal{N}$ if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$ or equivalently, as noted in [7], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. During the last years, the study of the commutativity of prime rings or 3-prime near-rings has been an active area of research. In this direction, in [5] H. E. Bell and G. Mason initialized this study using the notion of derivation defined in a prime ring. Let α be a map from $\mathcal{N}$ to $\mathcal{N}$, an additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is called an $(\alpha, 1)$-derivation if $d(xy) = d(x)\alpha(y) + xd(y)$ for all $x, y \in \mathcal{N}$. An additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is called an $(1, \alpha)$-derivation if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in \mathcal{N}$. Furthermore, an additive mapping $d\colon \mathcal{N} \to \mathcal{N}$ is called a two-sided α-derivation if d is an $(\alpha, 1)$-derivation as well as $(1, \alpha)$-derivation. Moreover, if d commutes with α, then d is called a semiderivation (see [6]). Clearly every semiderivation is a two-sided α-derivation, but the converse is not true. In case $\alpha = 1$, a two-sided α-derivation is just a derivation. But an example due to [1] proved that, we can find a two-sided α-derivation that is not a derivation. Hence, it should be interesting to study the commutativity of a near ring $\mathcal{N}$ admitting some differential conditions. Here we initiate the concepts of $(\alpha, 1)$-n-derivation, $(1, \alpha)$-n-derivation and α-n-derivation as follows:

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Definition 1.1. *Let $\mathcal{N}$ be a near-ring and let n be a fixed positive integer. An n-additive (i.e. additive in each argument) mapping $D\colon \underbrace{\mathcal{N}\times\mathcal{N}\times\cdots\times\mathcal{N}}_{n-times} \longrightarrow \mathcal{N}$ is called $(\alpha, 1)$-n-derivation if*

$$D(x_1y_1, x_2, x_3, \dots, x_n) = D(x_1, x_2, x_3, \dots, x_n)\alpha(y_1) + x_1D(y_1, x_2, x_3, \dots, x_n)$$

$$D(x_1, x_2y_2, x_3, \dots, x_n) = D(x_1, x_2, x_3, \dots, x_n)\alpha(y_2) + x_2D(x_1, y_2, x_3, \dots, x_n)$$

$$D(x_1, x_2, x_3, \dots, x_ny_n) = D(x_1, x_2, x_3, \dots, x_n)\alpha(y_n) + x_nD(x_1, x_2, x_3, \dots, y_n)$$

hold for all $x_i, y_i \in \mathcal{N}$ and $i \in \{1, 2, \dots, n\}$.

Similarly we can define a $(1, \alpha)$-n-derivation and a two sided α-n-derivation.

2 Some preliminaries

We begin with the following lemmas which are very interesting for developing the proofs of our main results.

Lemma 2.1 ([5], Lemma 1.1). Let $\mathcal{N}$ be a 3-prime near-ring.
(i) If $z \in Z(\mathcal{N}) - \{0\}$ and $xz \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.
(ii) If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.
(iii) If $\mathcal{N}$ admits a nonzero derivation d, then

$$(xd(y) + d(x)y)z = xd(y)z + d(x)yz \text{ for all } x, y, z \in \mathcal{N}\,.$$

(iv) If $z \in Z(\mathcal{N}) - \{0\}$, then z is not a zero divisor. Moreover, if $z + z \in Z(\mathcal{N})$, then $(\mathcal{N}, +)$ is an abelian group.

Lemma 2.2 ([5], Lemma 3 (iv)). Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a derivation d such that $d^2 = 0$, then $d = 0$.

Lemma 2.3. *Let $\mathcal{N}$ be a 3-prime near-ring, D a $(\alpha, 1)$-n-derivation associated with a map α and n is a positive integer. Then $\mathcal{N}$ satisfies the partial distributive law:*

$$\Big(D(x, x_2, \dots, x_n)\alpha(y) + xD(y, x_2, \dots, x_n)\Big)\alpha(z) = D(x, x_2, \dots, x_n)\alpha(yz) + xD(z, x_2, \dots, x_n)\alpha(z)$$

for all $x, y, z, x_i \in \mathcal{N}$, $i \in \{2, \dots, n\}$.

Proof. By definition of D, we have

$$\begin{aligned} D(xyz, x_2, \dots, x_n) &= D(xy, x_2, \dots, x_n)\alpha(z) + xyD(z, x_2, \dots, x_n) \\ &= \Big(D(x, x_2, \dots, x_n)\alpha(y) + xD(y, x_2, \dots, x_n)\Big)\alpha(z) + \\ &\quad xyD(z, x_2, \dots, x_n) \end{aligned}$$

On the other hand,

$$\begin{aligned} D(xyz, x_2, \ldots, x_n) &= D(x, x_2, \ldots, x_n)\alpha(yz) + xD(yz, x_2, \ldots, x_n) \\ &= D(x, x_2, \ldots, x_n)\alpha(yz) + xD(y, x_2, \ldots, x_n)\alpha(z) + \\ &\quad xyD(z, x_2, \ldots, x_n) \end{aligned}$$

Combining the above two equalities, we obtain that

$$\Big(D(x, x_2, \ldots, x_n)\alpha(y) + xD(y, x_2, \ldots, x_n)\Big)\alpha(z) = D(x, x_2, \ldots, x_n)\alpha(yz) + xD(y, x_2, \ldots, x_n)\alpha(z)\,. \qquad \square$$

Lemma 2.4 ([3, Lemma 1.3 (i) & Lemma 1.4 (i)]). *Let $\mathcal{N}$ be a 3-prime near-ring and U a nonzero semigroup ideal of $\mathcal{N}$.*
(i) If $Ux = \{0\}$, then $x = 0$.
(ii) If $x, y \in \mathcal{N}$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

Lemma 2.5 ([4, Theorem 2.9]). *Let $\mathcal{N}$ be a 3-prime near-ring. If U is a nonzero semigroup ideal of $\mathcal{N}$, then the following assertions are equivalent:*
(i) $[x, y] \in Z(\mathcal{N})$ for all $x, y \in U$.
(ii) $\mathcal{N}$ is a commutative ring.

Lemma 2.6. *Let $\mathcal{N}$ be a 3-prime near-ring, $U_1, U_2, \ldots, U_n$ nonzero semigroup ideals of $\mathcal{N}$ where n is a positive integer. If $\mathcal{N}$ admits a nonzero $(\alpha, 1)$-n-derivation D, then*
(i) $D(U_1, U_2, \ldots, U_n) \neq \{0\}$.
(ii) If $x \in \mathcal{N}$ and $xD(U_1, U_2, \ldots, U_n) = \{0\}$, then $x = 0$.

Proof. (i) Suppose that $D(u_1, u_2, \ldots, u_n) = 0$ for all $u_i \in U_i$. Replacing u_1 by u_1r_1 where $r_1 \in \mathcal{N}$, we get $0 = D(u_1r_1, u_2, \ldots, u_n) = D(u_1, u_2, \ldots, u_n)\alpha(r_1) + u_1D(r_1, u_2, \ldots, u_n)$ for all $u_i \in U_i, r_1 \in \mathcal{N}$ which implies that $u_1D(r_1, u_2, \ldots, u_n) = 0$ for all $u_i \in U_i, r_1 \in \mathcal{N}$, so $u_1\mathcal{N}D(r_1, u_2, \ldots, u_n) = \{0\}$ for all $u_i \in U_i, r_1 \in \mathcal{N}$. Since $U_1 \neq \{0\}$, then by the 3-primeness of $\mathcal{N}$, we obtain $D(r_1, u_2, \ldots, u_n) = 0$ for all $u_i \in U_i, r_1 \in \mathcal{N}, i \in \{2, \ldots, n\}$. Now proceeding inductively in a similar manner as above, it is obvious to see that $D(r_1, r_2, \ldots, r_n) = 0$ for all $r_1, r_2, .., r_n \in \mathcal{N}$. Hence, $D = 0$; a contradiction.
(ii) Assume that $xD(u_1, u_2, \ldots, u_n) = 0$ for all $u_i \in U_i$ and $i \in \{1, \ldots, n\}$. Putting s_1u_1 in place of u_1, where $s_1 \in U_1$ and using the definition of D, we arrive at $xs_1D(u_1, u_2, \ldots, u_n) = 0$ for all $u_i \in U_i, s_1 \in U_1$ and $i \in \{1, \ldots, n\}$. The above equation can be rewritten as $xU_1D(u_1, u_2, \ldots, u_n) = \{0\}$ for all $u_i \in U_i$ and $i \in \{1, \ldots, n\}$. By Lemma 2.4 (ii), we conclude that $x = 0$. $\square$

Lemma 2.7. *Let $\mathcal{N}$ be a 3-prime near-ring and U_1, U_2,..., U_n be nonzero semigroup ideals of $\mathcal{N}$ where n is a positive integer. If $\mathcal{N}$ admits a $(1, \alpha)$-n-derivation D associated with a map α, then*

$$D(x, x_2, \ldots, x_n)y + \alpha(x)D(y, x_2, \ldots, x_n) = \alpha(x)D(y, x_2, \ldots, x_n) + D(x, x_2, \ldots, x_n)y$$

for all $x, y \in U_1, x_i \in U_i, i \in \{2, \dots, n\}$.

Proof. By definition of D, we have

$$\begin{aligned} D(x(y+y), x_2, \dots, x_n) &= D(x, x_2, \dots, x_n)(y+y) + \alpha(x)D(y+y, x_2, \dots, x_n) \\ &= D(x, x_2, \dots, x_n)y + D(x, x_2, \dots, x_n)y \\ &+ \alpha(x)D(y, x_2, \dots, x_n) + \alpha(x)D(y, x_2, \dots, x_n)\,. \end{aligned}$$

for all $x, y \in U_1, x_i \in U_i, i \in \{2, \dots, n\}$.
By another way, we have

$$\begin{aligned} D(x(y+y), x_2, \dots, x_n) &= D(xy + xy, x_2, \dots, x_n) \\ &= D(xy, x_2, \dots, x_n) + D(xy, x_2, \dots, x_n) \\ &= D(x, x_2, \dots, x_n)y + \alpha(x)D(y, x_2, \dots, x_n) \\ &+ D(x, x_2, \dots, x_n)y + \alpha(x)D(y, x_2, \dots, x_n)\,. \end{aligned}$$

By comparing the last expressions, we conclude that

$$D(x, x_2, \dots, x_n)y + \alpha(x)D(y, x_2, \dots, x_n) = \alpha(x)D(y, x_2, \dots, x_n) + D(x, x_2, \dots, x_n)y$$

for all $x, y \in U_1, x_i \in U_i, i \in \{2, \dots, n\}$. □

In the year 1994 it was proved by Wang [7, Lemma 2] that if near-ring $\mathcal{N}$ admits a derivation d then $d(Z(\mathcal{N})) \subseteq Z(\mathcal{N})$. This result was extended for 3-prime near-rings with n-derivations in [2, Lemma 2.6] by M. Ashraf, M. Aslam Siddeeque and N. Parveen. Inspired by this result, we have generalized this result by using the notion of two sided α-n-derivation in 3-prime near-ring $\mathcal{N}$. More precisely, we have the following result:

Lemma 2.8. *Let $\mathcal{N}$ be a 3-prime near-ring and $U_2, \dots, U_n$ be nonzero semigroup ideals of $\mathcal{N}$ where $n \in \mathbb{N}^* - \{1\}$. If $\mathcal{N}$ admits a two sided α-n-derivation associated with a map α from $\mathcal{N}$ to $\mathcal{N}$ such that $\alpha(Z(\mathcal{N})) \subseteq Z(\mathcal{N})$, then $D\big(Z(\mathcal{N}) \times U_2 \times \cdots \times U_n\big) \subseteq Z(\mathcal{N})$.*

Proof. Let $z \in Z(\mathcal{N})$, we have $D(zr, u_2, \dots, u_n) = D(rz, u_2, \dots, u_n)$ for all $r \in \mathcal{N}$, $u_i \in U_i$ and $i \in \{2, \dots, n\}$. By definition of D and Lemma 2.7, we obtain

$$\alpha(z)D(r, u_2, \dots, u_n) + D(z, u_2, \dots, u_n)r = D(r, u_2, \dots, u_n)\alpha(z) + rD(z, u_2, \dots, u_n)\,.$$

Since $\alpha(Z(\mathcal{N})) \subseteq Z(\mathcal{N})$, the last equation gives us $D(z, u_2, \dots, u_n)r = rD(z, u_2, \dots, u_n)$ for all $u_2, \dots, u_n \in U_i$ for $i = 2, \dots, n$. i.e.; $D\big(Z(\mathcal{N}) \times U_2 \times \cdots \times U_n\big) \subseteq Z(\mathcal{N})$. □

3 Main Results

Theorem 3.1. *Let $\mathcal{N}$ be a 3-prime near-ring and $U_1, U_2, \dots, U_n$ nonzero semigroup ideals of $\mathcal{N}$ where n is a positive integer. If $\mathcal{N}$ admits a nonzero two sided α-n-derivation D such that $D(U_1 \times U_2 \times \cdots \times U_n) \subseteq Z(\mathcal{N})$, where α is a map of $\mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Proof. By hypothesis, we have

$$D(u_1, u_2, \ldots, u_n) \in Z(\mathcal{N}) \text{ for all } u_i \in U_i, i \in \{1, 2, \ldots, n\}\,. \tag{1}$$

Replacing u_1 by $v_1 u_1$ where $v_1 \in \mathcal{N}$ in (1) and using Lemma 2.3, we arrive at

$$D(v_1, u_2, \ldots, u_n)\alpha(u_1 z) + v_1 D(u_1, u_2, \ldots, u_n)\alpha(z) = \alpha(z)D(v_1, u_2, \ldots, u_n)\alpha(u_1) + \alpha(z)v_1 D(u_1, u_2, \ldots, u_n) \tag{2}$$

for all $v_1, z \in \mathcal{N}$, $u_i \in U_i$, $i \in \{1, \ldots, n\}$. Replacing v_1 by $\alpha(z)$ in (2), we obtain for all $z \in \mathcal{N}$, $u_i \in U_i$, $i \in \{1, \ldots, n\}$

$$D(\alpha(z), u_2, \ldots, u_n)\alpha(u_1 z) = \alpha(z)D(\alpha(z), u_2, \ldots, u_n)\alpha(u_1)\,. \tag{3}$$

Then (3) can be rewritten as

$$D(\alpha(z), u_2, \ldots, u_n)\Big(\alpha(u_1 z) - \alpha(z)\alpha(u_1)\Big) = 0 \text{ for all } z \in \mathcal{N}, u_i \in U_i, i \in \{1, \ldots, n\}\,. \tag{4}$$

Which means that

$$\Big(\alpha(u_1 z) - \alpha(z)\alpha(u_1)\Big)\mathcal{N}D\big(\alpha(z), u_2, \ldots, u_n\big) = \{0\} \text{ for all } z \in \mathcal{N}, u_i \in U_i,\ i \in \{1, \ldots, n\}\,. \tag{5}$$

In light of the 3-primeness of $\mathcal{N}$, (5) implies that

$$D(\alpha(z), u_2, \ldots, u_n) = 0 \text{ or } \alpha(u_1 z) = \alpha(z)\alpha(u_1) \tag{6}$$

for all $z \in \mathcal{N}$, $u_i \in U_i$, $i \in \{1, \ldots, n\}$.
If there exists an element z_1 of $\mathcal{N}$ such that $D(\alpha(z_1), u_2, \ldots, u_n) = 0$ for all $u_i \in U$, $i \in \{1, \ldots, n\}$, then (2) gives us for all $z \in \mathcal{N}$, $u_i \in U_i$, $i \in \{1, \ldots, n\}$

$$\alpha(z_1)D(u_1, u_2, \ldots, u_n)\alpha(z) = \alpha(z)\alpha(z_1)D(u_1, u_2, \ldots, u_n) \tag{7}$$

this further implies that

$$[\alpha(z), \alpha(z_1)]\mathcal{N}D\big(u_1, u_2, \ldots, u_n\big) = \{0\} \text{ for all } z \in \mathcal{N}, u_i \in U_i, i \in \{1, \ldots, n\}\,. \tag{8}$$

By the 3-primeness of $\mathcal{N}$, we obtain $\alpha(z)\alpha(z_1) = \alpha(z_1)\alpha(z)$ for all $z \in \mathcal{N}$. In this case, replacing v_1 by $\alpha(v_1)$ and z by z_1 in (2), we arrive at

$$\Big(\alpha(u_1 z_1) - \alpha(z_1)\alpha(u_1)\Big)\mathcal{N}D\big(\alpha(v_1), u_2, \ldots, u_n\big) = \{0\}\,. \tag{9}$$

for all $v_1 \in \mathcal{N}$, $u_i \in U_i$, $i \in \{1, \ldots, n\}$.
Using the 3-primeness of $\mathcal{N}$, we find that
$D(\alpha(v_1), u_2, \ldots, u_n) = 0$ or $\alpha(u_1 z_1) = \alpha(z_1)\alpha(u_1)$ for all $v_1 \in \mathcal{N}$, $u_i \in U_i$.
In this case, (6) becomes

$$D(\alpha(v_1), u_2, \ldots, u_n) = 0 \text{ or } \alpha(u_1 z) = \alpha(z)\alpha(u_1) \text{ for all } z, v_1 \in \mathcal{N}, u_i \in U_i \tag{10}$$

(i) Suppose that $D(\alpha(v_1), u_2, \ldots, u_n) = 0$ for all $v_1 \in \mathcal{N}$, $u_i \in U_i$. Replacing u_1 by $\alpha(v_1)u_1$ in (1) and using the definition of D, we get

$$\alpha(v_1)D(u_1, u_2, \ldots, u_n) \in Z(\mathcal{N}) \text{ for all } v_1 \in \mathcal{N}, u_i \in U_i\,. \tag{11}$$

According to Lemma 2.6 (i), (11) implies that $\alpha(v_1) \in Z(\mathcal{N})$ for all $v_1 \in \mathcal{N}$.
(ii) If $\alpha(u_1 z) = \alpha(z)\alpha(u_1)$ for all $u_1 \in U_1$, $z \in \mathcal{N}$. Using (2) and Lemma 2.6 (i), we can easily show that $\alpha(z) \in Z(\mathcal{N})$ for all $z \in \mathcal{N}$. Therefore, according to two previous cases we arrive at $\alpha(z) \in Z(\mathcal{N})$ for all $z \in \mathcal{N}$. Since D is a $(\alpha, 1)$-n-derivation, then for all $u_i \in U_i$, $v_1 \in U_1$, $i \in \{1, \ldots, n\}$, we have

$$D(u_1, u_2, \ldots, u_n)\alpha(v_1) + u_1 D(v_1, u_2, \ldots, u_n) \in Z(\mathcal{N})\,. \tag{12}$$

Since $\alpha(v_1) \in Z(\mathcal{N})$ and $D(u_1, u_2, \ldots, u_n) \in Z(\mathcal{N})$, then (12) yields

$$[v_1, r]\mathcal{N}D(u_1, u_2, \ldots, u_n) = \{0\} \text{ for all } r \in \mathcal{N}, u_i \in U_i, v_1 \in U_1, i \in \{1, 2, \ldots, n\}\,. \tag{13}$$

Since $\mathcal{N}$ is 3-prime and $D \neq 0$, then $\mathcal{N}$ is a commutative ring. □

Theorem 3.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and U, U_1, U_2,..., U_n nonzero semigroup ideals of $\mathcal{N}$ where n is a positive integer. If $\mathcal{N}$ admits a nonzero derivation d and a nonzero two sided α-n-derivation D associated with a map α such that $d(U)D(U_1 \times U_2 \times \cdots \times U_n) = D(U_1 \times U_2 \times \cdots \times U_n)d(U)$, then $\mathcal{N}$ is a commutative ring.*

Proof. we are assuming that

$$d(u)D(u_1, u_2, \ldots, u_n) = D(u_1, u_2, \ldots, u_n)d(u) \tag{14}$$

for all $u \in U$, $u_i \in U_i$, $i \in \{1, \ldots, n\}$.
Replacing u by uv where $u \in \mathcal{N}$ in (14) and using Lemma 2.1 (iii), we obtain

$$\begin{aligned} d(u)vD(u_1, u_2, \ldots, u_n) + ud(v)D(u_1, u_2, \ldots, u_n) = D(u_1, u_2, \ldots, u_n)d(u)v \\ + D(u_1, u_2, \ldots, u_n)ud(v)\,. \end{aligned}$$

Substituting $d(u)$ for u where $u \in U$, we get

$$d^2(u)vD(u_1, u_2, \ldots, u_n) = d^2(u)D(u_1, u_2, \ldots, u_n)v \tag{15}$$

for all $u, v \in U$, $u_i \in U_i$, $i \in \{1, \ldots, n\}$.
Putting vt in place of v where $t \in \mathcal{N}$ in (15) and using it again, we arrive at

$$d^2(u)v[t, D(u_1, u_2, \ldots, u_n)] = 0 \text{ for all } v \in U, u_i \in U_i, u, t \in \mathcal{N}, i \in \{1, \ldots, n\}\,.$$

This equation can be rewritten as

$$d^2(u)U[t, D(u_1, u_2, \ldots, u_n)] = \{0\} \text{ for all } u, t \in \mathcal{N}, u_i \in U_i, i \in \{1, \ldots, n\}\,.$$

By Lemma 2.4 (ii), the latter relation implies

$$d^2(u) = 0 \text{ or } [t, D(u_1, u_2, \dots, u_n)] = 0 \text{ for all } u, t \in \mathcal{N}, u_i \in U_i, i \in \{1, \dots, n\}.$$

Since $d \neq 0$, then by Lemma 2.2 the first condition can't occur; in this case, we arrive at $D(U_1 \times U_2 \times \cdots \times U_n) \subseteq Z(\mathcal{N})$ and by application of Theorem 3.1, we obtain the desired conclusion. □

Theorem 3.3. *Let $\mathcal{N}$ be a 3-prime near-ring and $U_1, U_2, \dots, U_n$ nonzero semigroup ideals of $\mathcal{N}$ where n is a positive integer. If $\mathcal{N}$ admits a nonzero two sided α-n-derivation D associated with a map α such that $\alpha(Z(\mathcal{N})) \subseteq Z(\mathcal{N})$ and $D([x, y], x_2, \dots, x_n) = 0$ for all $x, y \in U_1, x_i \in U_i, i \in \{2, \dots, n\}$, then $\mathcal{N}$ is a commutative ring.*

Proof. We are given that

$$D([x, y], x_2, \dots, x_n) = 0 \quad \text{for all } x, y \in U_1, x_i \in U_i, i \in \{2, \dots, n\}. \tag{16}$$

Putting xy instead of y in (16), then for all $x, y \in U_1, x_i \in U_i, i \in \{2, \dots, n\}$ we have

$$D(x, x_2, \dots, x_n)[x, y] + \alpha(x)D([x, y], x_2, \dots, x_n) = 0.$$

Using (16), the above relation implies

$$D(x, x_2, \dots, x_n)xy = D(x, x_2, \dots, x_n)yx \quad \text{for all } x, y \in U_1, x_i \in U_i, i \in \{2, \dots, n\}. \tag{17}$$

Replacing y by yt where $t \in \mathcal{N}$ in (17) and using it again, we obtain

$$D(x, x_2, \dots, x_n)U_1[x, t] = \{0\} \quad \text{for all } t \in \mathcal{N}, x \in U_1, x_i \in U_i, i \in \{2, \dots, n\}. \tag{18}$$

3-primeness of $\mathcal{N}$ gives

$$D(x, x_2, \dots, x_n) = 0 \text{ or } x \in Z(\mathcal{N}) \quad \text{for all } x \in U_1, x_i \in U_i, i \in \{2, \dots, n\}.$$

Using Lemma 2.8, the both cases reduce to $D(x, x_2, \dots, x_n) \in Z(\mathcal{N})$ for all $x \in U_1, x_i \in U_i, i \in \{2, \dots, n\}$, so $D(U_1 \times \cdots \times U_n) \subseteq Z(\mathcal{N})$. By Theorem 3.1, we conclude that $\mathcal{N}$ is a commutative ring. □

Theorem 3.4. *Let $\mathcal{N}$ be a 3-prime near-ring and $U, U_1, U_2, \dots, U_n$ be nonzero semigroup ideals of $\mathcal{N}$ where n is a positive integer. If $\mathcal{N}$ admits a nonzero two sided α-n-derivation D associated with a map α. Then the following assertions are equivalent:*
(i) $[D(x_1, x_2, \dots, x_n), y] \in Z(\mathcal{N})$ for all $x_i \in U_i, i \in \{1, \dots, n\}, y \in U$.
(ii) $\mathcal{N}$ is a commutative ring.

Proof. It is obvious that $(ii) \Rightarrow (i)$.
$(i) \Rightarrow (ii)$. We may now suppose that

$$[D(x_1, x_2, \dots, x_n), y] \in Z(\mathcal{N}) \quad \text{for all } x_i \in U_i, i \in \{1, \dots, n\}, y \in U. \tag{19}$$

Replacing y by $D(x_1, x_2, \ldots, x_n)y$ in (19) and using it again, we obtain

$$D(x_1, x_2, \ldots, x_n)[D(x_1, x_2, \ldots, x_n), y] \in Z(\mathcal{N}) \quad \text{for all} \quad x_i \in U_i, i \in \{1, \ldots, n\}, y \in U. \tag{20}$$

By Lemma 2.1 (i), (20) implies

$$D(x_1, x_2, \ldots, x_n) \in Z(\mathcal{N}) \text{ or } D(x_1, x_2, \ldots, x_n)y = yD(x_1, x_2, \ldots, x_n) \tag{21}$$

for all $x_i \in U_i, i \in \{1, \ldots, n\}, y \in U$.

Suppose there are $x_i \in U_i, i \in \{1, \ldots, n\}$ such that $D(x_1, x_2, \ldots, x_n)y = yD(x_1, x_2, \ldots, x_n)$ for all $y \in U$. Replacing y by yt where $t \in \mathcal{N}$, we obtain

$$\begin{aligned} ytD(x_1, x_2, \ldots, x_n) &= D(x_1, x_2, \ldots, x_n)yt \\ &= yD(x_1, x_2, \ldots, x_n)t \quad \text{for all} \quad y \in U, t \in \mathcal{N} \end{aligned}$$

which reduces to

$$U[D(x_1, x_2, \ldots, x_n), t] = \{0\} \quad \text{for all} \quad t \in \mathcal{N}\,. \tag{22}$$

From equation (21) it follows at once by Lemma 2.4 (i) that $D(x_1, x_2, \ldots, x_n) \in Z(\mathcal{N})$ for all $x_i \in U_i, i \in \{1, \ldots, n\}$ and Theorem 3.1 forces that $\mathcal{N}$ is a commutative ring. □

Theorem 3.5. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero two sided α-n-derivation D associated with a nonzero map α such that $D([x, y], x_2, \ldots, x_n) = [x, y]$ for all $x, y, x_i \in \mathcal{N}, i \in \{2, \ldots, n\}$, then $\mathcal{N}$ is a commutative ring.*

Proof. First we consider the case

$$D([x, y], x_2, \ldots, x_n) = [x, y] \quad \text{for all} \quad x, y, x_i \in \mathcal{N}, i \in \{2, \ldots, n\}\,. \tag{23}$$

Taking xy instead of y in (23), we have

$$D(x, x_2, \ldots, x_n)[x, y] + \alpha(x)D([x, y], x_2, \ldots, x_n) = x[x, y]\,.$$

for all $x, y, x_i \in \mathcal{N}, i \in \{2, \ldots, n\}$.
Using (23), we obtain

$$D(x, x_2, \ldots, x_n)[x, y] + \alpha(x)[x, y] = x[x, y]\,. \tag{24}$$

for all $x, y, x_i \in \mathcal{N}, i \in \{2, \ldots, n\}$.
Replacing x by $[u, v]$ in (24), we obtain

$$\alpha([u, v])[u, v]y = \alpha([u, v])y[u, v] \quad \text{for all } u, v, y \in \mathcal{N}\,. \tag{25}$$

Substituting yz for y in (25), we get

$$\alpha([u, v])yz[u, v] = \alpha([u, v])y[u, v]z \quad \text{for all} \quad u, v, y, z \in \mathcal{N}$$

and therefore $\alpha([u,v])y[[u,v],z] = 0$ which can be rewritten as

$$\alpha([u,v])\mathcal{N}[[u,v],z] = \{0\} \quad \text{for all} \quad u,v,z \in \mathcal{N}\,. \tag{26}$$

By the 3-primeness of $\mathcal{N}$, (26) becomes

$$\alpha([u,v]) = 0 \quad \text{or} \quad [u,v] \in Z(\mathcal{N}) \quad \text{for all} \quad u,v \in \mathcal{N}\,. \tag{27}$$

Suppose there exist $u_0, v_0 \in \mathcal{N}$ such that $\alpha([u_0,v_0]) = 0$, we have

$$D([u_0,v_0]xy, x_2, \ldots, x_n) = [u_0,v_0]xy \quad \text{for all} \quad x,y,x_i \in \mathcal{N}, i \in \{2,\ldots,n\}\,.$$

On the other hand,

$$\begin{aligned} D([u_0,v_0]xy, x_2, \ldots, x_n) &= D([u_0,v_0]x, x_2, \ldots, x_n)\alpha(y) + [u_0,v_0]xD(y, x_2, \ldots, x_n) \\ &= [u_0,v_0]x\alpha(y) + [u_0,v_0]xD(y, x_2, \ldots, x_n)\,. \end{aligned}$$

for all $x, y, x_i \in \mathcal{N}$, for all $i \in \{1, 2, \ldots, n\}$.
The above relation yields that

$$[u_0,v_0]x\alpha(y) + [u_0,v_0]xD(y, x_2, \ldots, x_n) = [u_0,v_0]xy \quad \text{for all} \quad x,y \in \mathcal{N}\,. \tag{28}$$

Substituting $[r,s]$ for y in (28), we obtain

$$[u_0,v_0]\mathcal{N}\alpha([r,s]) = \{0\} \quad \text{for all} \quad r,s \in \mathcal{N}$$

Using the 3-primeness of $\mathcal{N}$, we obtain $[u_0,v_0] = 0$ or $\alpha([r,s]) = 0$ for all $r,s \in \mathcal{N}$ in this case, (27) becomes

$$\alpha([r,s]) = 0 \quad \text{or} \quad [u,v] \in Z(\mathcal{N}) \quad \text{for all} \quad u,v,r,s \in \mathcal{N}\,. \tag{29}$$

If $\alpha([r,s]) = 0$ for all $r,s \in \mathcal{N}$, by calculation of $D([r,s]xy, x_2, \ldots, x_n)$ by two different ways, we can easily arrive at

$$[r,s]x(\alpha(y) + D(y, x_2, \ldots, x_n) - y) = 0 \quad \text{for all} \quad r,s,x,y \in \mathcal{N}$$

which reduces to

$$[r,s]\mathcal{N}(\alpha(y) + D(y, x_2, \ldots, x_n) - y) = \{0\} \quad \text{for all} \quad r,s,y,x_i \in \mathcal{N}, i \in \{2,\ldots,n\}\,. \tag{30}$$

By the 3-primeness of $\mathcal{N}$, (30) shows that

$$[r,s] = 0 \quad \text{or} \quad \alpha(y)+D(y, x_2, \ldots, x_n)-y = 0 \quad \text{for all} \quad r,s,y,x_i \in \mathcal{N}, i \in \{2,\ldots,n\}. \tag{31}$$

Now, suppose that $\alpha(y) + D(y, x_2, \ldots, x_n) = y$ for all $y, x_i \in \mathcal{N}$, $i \in \{2,\ldots,n\}$. Replacing y by yt, then $\alpha(yt)+D(yt, x_2, \ldots, x_n) = yt$ for all $y, t, x_i \in \mathcal{N}$, $i \in \{2,\ldots,n\}$, which implies that $\alpha(yt) + D(y, x_2, \ldots, x_n)\alpha(t) + yD(t, x_2, \ldots, x_n) = yt$ for all $y, t, x_i \in \mathcal{N}$, $i \in \{2,\ldots,n\}$, so $\alpha(yt) + D(y, x_2, \ldots, x_n)\alpha(t) + y(-\alpha(t) + t) = yt$ for all $y, t, x_i \in \mathcal{N}$,

$i \in \{2, \dots, n\}$, which can be rewritten $D(y, x_2, \dots, x_n)\alpha(t) = -\alpha(yt) + y\alpha(t)$ for all $y, t, x_i \in \mathcal{N}$, $i \in \{2, \dots, n\}$. Replacing y by ys in the last expression and using it again with Lemma 2.3, we get $D(y, x_2, \dots, x_n)\alpha(st) + yD(s, x_2, \dots, x_n)\alpha(t) = -\alpha(yst) + ys\alpha(t)$ for all $y, s, t, x_i \in \mathcal{N}$, $i \in \{2, \dots, n\}$, which implies that $-\alpha(yst) + y\alpha(st) + y(-\alpha(st) + s\alpha(t)) = -\alpha(yst) + ys\alpha(t)$ for all $y, s, t \in \mathcal{N}$ which gives $y(-\alpha(st) + s\alpha(t)) = 0$ for all $y, s, t \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$, we arrive at $\alpha(st) = s\alpha(t)$ for all $y, s, t \in \mathcal{N}$. According to our hypothesis, we have $\alpha([r, s]y) + D([r, s]y, x_2, \dots, x_n) = [r, s]y$ for all $y, r, s, x_i \in \mathcal{N}$, $i \in \{2, \dots, n\}$, then $\alpha([r, s]y) + D([r, s], x_2, \dots, x_n)y = [r, s]y$ for all $y, r, s, x_i \in \mathcal{N}$, $i \in \{2, \dots, n\}$, which forces that $\alpha([r, s]y) = 0$ for all $y, r, s \in \mathcal{N}$. Since $\alpha(st) = s\alpha(t)$ for all $s, t \in \mathcal{N}$, the last equation becomes $[r, s]\alpha(y) = 0$ for all $r, s, y \in \mathcal{N}$ and replacing y by zy, we arrive at $[r, s]\mathcal{N}\alpha(y) = \{0\}$ for all $r, s, y \in \mathcal{N}$. Since $\alpha \neq 0$ and $\mathcal{N}$ is 3-prime, then $[r, s] = 0$ for all $r, s \in \mathcal{N}$. In all cases, (27) becomes $[u, v] \in Z(\mathcal{N})$ for all $u, v \in \mathcal{N}$ and Lemma 2.5 forces that $\mathcal{N}$ is a commutative ring. □

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Luisa Carini and Vincenzo De Filippis

Generalized Skew Derivations satisfying the second Posner's theorem on Lie ideals

Abstract: Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F is a generalized skew derivation of R and L is a non-central Lie ideal of R. If

$$\Big[[F(u), u], F(u)\Big] = 0$$

for all $u \in L$, then either there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$ or R satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree 4, and there exist $a \in Q_r$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$, for all $x \in R$.

Keywords: Generalized skew derivation; Lie ideal; Prime Ring.

1 Introduction

Let R be a prime ring with center $Z(R)$ and d be a nonzero derivation of R. The well-known theorem of Posner [29] states that if $[d(x), x] \in Z(R)$ for all $x \in R$, then R must be commutative. This theorem indicates how the global structure of a ring R is closely related to the behaviour of additive mappings defined on R. Following this line of investigation, several authors have generalized the Posner's Theorem in various directions and studied the relationship between the structure of a prime ring R and the behavior of some additive mappings f which satisfies the condition $[f(x), x] \in Z(R)$ (or $[f(x), x] = 0$). For instance, Posner's theorem is extended to the case of Lie ideals of prime rings by Lanski in [23]. In [27] Mayne proves an analogous af Posner's result for automorphisms which are centralizing on prime rings. He also extends the result to Lie ideals as follows: if R has characteristic not equal to two, σ is an automorphism of R and L is a Lie ideal of R such that σ is nontrivial on L and $[\sigma(x), x] \in Z(R)$, for every $x \in L$, then $L \subseteq Z(R)$ ([28]).

Many researchers in this area have analyzed in detail the case when derivations and automorphisms are replaced by other additive maps acting on two-sided ideals, one-sided ideals and Lie ideals of prime and semiprime rings. Let $F\colon R \to R$ be an additive map on the prime ring R, defined as follows: $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$ and for some derivations d of R. Such mapping F is called *generalized derivations* of R with associated derivation d. Obviously, any derivation of R and any mapping of R with the form $f(x) = ax + xb$ for some $a, b \in R$, are both generalized derivations. The

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latter are usually called *inner generalized derivations* and have been primarily studied on operator algebras, therefore any investigation from the algebraic point of view might be interesting.

In two recent papers the second author has extended the Posner's theorem for generalized derivation as follows:

Theorem 1.1 ([15]). *Let R be a prime ring of characteristic different from 2, C the extended centroid of R, and F a generalized derivations of R. If $[[F(x), x], F(x)] = 0$ for all $x \in R$ then either R is commutative or $F = \lambda x$, for all $x \in R$ and some $\lambda \in C$.*

Theorem 1.2 ([16]). *Let R be a prime ring of characteristic different from 2, $Z(R)$ its center, U its Utumi quotient ring, C its extended centroid, F a non-zero generalized derivation of R, L a non-central Lie ideal of R. If $\Big[[F(u), u], F(u)\Big] \in Z(R)$ for all $u \in L$ then one of the following holds:*

1. *there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*
2. *R satisfies the standard identity $s_4(x_1, \dots, x_4)$ and there exist $a \in U$, $\alpha \in C$ such that $F(x) = ax + xa + \alpha x$, for all $x \in R$.*

Furthermore, in [19], Argaç and Demir provide a one-sided version of the results contained in [15, 16], extending the argument to right ideals of prime rings.

Following this line of investigations, in the current article we will study the set

$$S = \left\{ \Big[[F(u), u], F(u)\Big] | u \in L \right\},$$

where F is a generalized skew derivation and L is a non-central Lie ideal of R. We will now recall the definition of generalized skew derivations of R. Let R be an associative ring and α be an automorphism of R. An additive mapping $d \colon R \longrightarrow R$ is called a *skew derivation* of R if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in R$ and α is called an *associated automorphism* of d. An additive mapping $G \colon R \longrightarrow R$ is said to be a *generalized skew derivation* of R if there exists a skew derivation d of R with associated automorphism α such that

$$G(xy) = G(x)y + \alpha(x)d(y)$$

for all $x, y \in R$, d is said to be an *associated skew derivation* of G and α is called an *associated automorphism* of G. Any mapping of R with form $G(x) = ax + \alpha(x)b$ for some $a, b \in R$ and $\alpha \in Aut(R)$, is called *inner generalized skew derivation*. In particular, if $a = -b$, then G is called *inner skew derivation*. If a generalized skew derivation (respectively, a skew derivation) is not inner, then it is usually called *outer*.

The concept of generalized skew derivation unifies the notions of skew derivation and generalized derivation, which have been investigated by many researchers from various points of view (see [3, 4, 5, 6, 7, 8, 9, 18, 24, 25]).

In what follows, let Q_r be the right Martindale quotient ring of R, Q be the two-sided Martindale quotient ring of R and $C = Z(Q) = Z(Q_r)$ be the center of Q and Q_r. C is usually called the *extended centroid* of R and is a field when R is a prime ring. It should be remarked that Q is a centrally closed prime C-algebra. We refer the reader to [2] for the definitions and the related properties of these objects.

It is well known that automorphisms, derivations and skew derivations of R can be extended both to Q and Q_r. In [3] Chang extends the definition of generalized skew derivation to the right Martindale quotient ring Q_r of R as follows: by a (right) generalized skew derivation we mean an additive mapping $G\colon Q_r \longrightarrow Q_r$ such that $G(xy) = G(x)y + \alpha(x)d(y)$ for all $x, y \in Q_r$, where d is a skew derivation of R and α is an automorphism of R. Moreover, there exists $G(1) = a \in Q_r$ such that $G(x) = ax + d(x)$ for all $x \in R$. Furthermore, if $G(1) \in Q$, then G can be extended to Q.

The main result of this article is

Theorem 1.3. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F is a generalized skew derivation of R and L is a non-central Lie ideal of R. If*

$$\Big[[F(u), u], F(u)\Big] = 0$$

for all $u \in L$, then either there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$ or R satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree 4, and there exist $a \in Q_r$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$, for all $x \in R$.

We will fix some notations and collect some existing results which will be used in the sequel.

Let $SDer(Q)$ be the set of all skew-derivations of Q. By a *skew-derivation word* we mean an additive mapping Δ of the form $\Delta = d_1 d_1 \dots d_m$, where $d_i \in SDer(Q)$. A *skew-differential polynomial* is a generalized polynomial with coefficients in Q of the form $\Phi(\Delta_j(x_i))$ involving noncommutative indeterminates x_i on which the derivation words Δ_j act as unary operations. The skew-differential polynomial $\Phi(\Delta_j(x_i))$ is said to be a *skew-differential identity* on a subset T of Q if it vanishes on any assignment of values from T to its indeterminates x_i.

We refer the reader to [10, 11, 12, 13] for a complete description of these objects. In particular we recall the following result:

Fact 1.1. In [14] Chuang and Lee investigate polynomial identities with skew derivations. More precisely in [14, Theorem 1] they prove that if D is an outer skew derivation of R which satisfies the generalized polynomial identity $\Phi(x_i, D^k(x_j))$, then:

1. If D is not left-algebraic modulo X-inner skew derivations, then R satisfies the generalized polynomial identity $\Phi(x_i, y_{kj})$, where x_i and y_{kj} are distinct indeterminates.

2. If D is algebraic modulo X-inner skew derivations such that the minimal order m of such algebraic dependence is strictly bigger than k, then R satisfies the generalized polynomial identity $\Phi(x_i, y_{kj})$, where x_i and y_{kj} are distinct indeterminates.

As a consequence of this result, we would like to point out that, if $k = 1$, that is $\Phi(x_i, D(x_j))$ is a generalized polynomial identity for R, then, in any case, $\Phi(x_i, y_j)$ is also a generalized polynomial identity for R, where x_i and y_j are distinct indeterminates.

Fact 1.2. Let R be a prime ring and I be a two-sided ideal of R. Then I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [10]). Furthermore, I, R, and Q_r satisfy the same generalized polynomial identities with automorphisms (see [12, Theorem 1]).

2 The case of inner generalized skew derivations with associated inner automorphisms

In this section we will prove the following:

Proposition 2.1. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F is an additive of R defined as follows:*

$$F(x) = ax + qxp$$

for all $x \in R$ and suitable fixed $a, q, p \in Q_r$, with invertible element q of Q_r. If

$$\Big[[F(u), u], F(u)\Big] = 0$$

for all $u \in [R, R]$, then one of the following holds:

1. *$a, q, p \in C$*
2. *$p \in C$ and $a + qp \in C$;*
3. *R satisfies $s_4(x_1, \ldots, x_4)$, $q \in C$ and $a - qp \in C$.*

In this section we always assume that R satisfies the following generalized polynomial identity

$$\begin{aligned}&\Psi(x_1, x_2) =\\ &\Big[a[x_1, x_2]^2 + q[x_1, x_2]p[x_1, x_2] -\\ &[x_1, x_2]a[x_1, x_2] - [x_1, x_2]q[x_1, x_2]p, a[x_1, x_2] + q[x_1, x_2]p\Big].\end{aligned} \tag{1}$$

Remark 2.1. *We would like to point out that in the following Lemmas (2.1, 2.2, 2.3, 2.4 and 2.5), the element $q \in Q_r$ is fixed but not necessarily invertible.*

We begin with:

Lemma 2.1. *Let $R = M_m(C)$, $m \geq 3$ and denote $q = \sum_{hl} q_{hl}e_{hl}$ and $p = \sum_{hl} p_{hl}e_{hl}$ for $q_{hl}, p_{hl} \in C$. Then either $q \in Z(R)$ or $p \in Z(R)$.*

Proof. Let $i \neq j$ and $[r_1, r_2] = e_{ij} \neq 0$ in (1). We notice that the (j, i)-entry of $\Psi(r_1, r_2)$ is

$$2q_{ji}^2 p_{ji}^2 = 0\,,$$

which implies $q_{ji}p_{ji} = 0$, because R is 2-torsion free. Moreover, for any automorphism $\varphi\colon R \to R$, it is easy to check that $\varphi(a)$, $\varphi(q)$, $\varphi(p)$ satisfy the same condition (1) as a, q, p. Therefore, by Proposition 1 in [17], it follows that either $q \in Z(R)$ or $p \in Z(R)$, as required. □

Lemma 2.2. *Let $R = M_2(C)$ and let $a = \sum_{hl} a_{hl}e_{hl}$, $p = \sum_{hl} p_{hl}e_{hl}$, $q = \sum_{hl} q_{hl}e_{hl}$, for $a_{hl}, q_{hl}, p_{hl} \in C$. Then either $q \in Z(R)$ or $p \in Z(R)$.*

Proof. As above, choose $[r_1, r_2] = e_{ij}$ in (1), then

$$q_{ji}p_{ji} = 0\,. \tag{2}$$

Our first aim is to show that either q is a diagonal matrix or p is a diagonal one.

Here we assume that both q and p are not diagonal, without loss of generality suppose $p_{21} \neq 0$. Thus, by (2), it follows that $q_{21} = 0$. Moreover, since q is not diagonal, we also have $q_{12} \neq 0$, so that, again by (2), $p_{12} = 0$ follows.

In particular we have that, for $[r_1, r_2] = e_{21}$ in (1),

$$\Big[qe_{21}pe_{21} - e_{21}ae_{21} - e_{21}qe_{21}p,\, ae_{21} + qe_{21}p\Big] = 0 \tag{3}$$

and for $[r_1, r_2] = e_{12}$ in (1),

$$\Big[qe_{12}pe_{12} - e_{12}ae_{12} - e_{12}qe_{12}p,\, ae_{12} + qe_{12}p\Big] = 0\,. \tag{4}$$

Notice that, the $(2, 1)$-entry of the matrix (3) is $-(a_{12} + q_{12}p_{11})^2$, that is

$$a_{12} + q_{12}p_{11} = 0\,. \tag{5}$$

Moreover the $(1, 2)$-entry of the matrix (4) is $(a_{21} - q_{11}p_{21}) \cdot (q_{11}p_{21} - a_{21})$, that is

$$a_{21} - q_{11}p_{21} = 0\,. \tag{6}$$

By using (5) and (6) and for $[r_1, r_2] = e_{12} + e_{21}$ in (1) we get

$$\begin{bmatrix} 0 & 2\alpha \cdot (a_{11} + q_{11}p_{22}) \\ -2\alpha \cdot (a_{22} + q_{22}p_{11}) & 0 \end{bmatrix} = 0$$

where $\alpha = a_{11} - a_{22} + q_{11}p_{22} - p_{11}q_{22}$. Hence, either $a_{11} - a_{22} + q_{11}p_{22} - p_{11}q_{22} = 0$, or both $a_{11} + q_{11}p_{22} = 0$ and $a_{22} + q_{22}p_{11} = 0$. It is easy to see that, in any case, $a_{22} - a_{11} - q_{11}p_{22} + q_{22}p_{11} = 0$. The previous argument says that:

$$\text{if} \quad p_{12} = 0 \quad \text{and} \quad q_{21} = 0 \quad \text{then} \quad a_{22} - a_{11} - q_{11}p_{22} + q_{22}p_{11} = 0\,. \tag{7}$$

Let $\varphi(x) = (1+e_{12})x(1-e_{12}) \in Aut(M_2(C))$ and $\chi(x) = (1-e_{12})x(1+e_{12}) \in Aut(M_2(C))$. Note that the elements $\varphi(a)$, $\varphi(q)$, $\varphi(p)$, $\chi(a)$, $\chi(q)$, $\chi(p)$ satisfy the same conditions of a, q, p. Denote $\varphi(a) = \sum_{hl} a'_{hl}e_{hl}$, $\chi(a) = \sum_{hl} a''_{hl}e_{hl}$, $\varphi(q) = \sum_{hl} q'_{hl}e_{hl}$, $\chi(q) = \sum_{hl} q''_{hl}e_{hl}$, $\varphi(p) = \sum_{hl} p'_{hl}e_{hl}$, $\chi(p) = \sum_{hl} p''_{hl}e_{hl}$, for $a'_{hl}, p'_{hl}, q'_{hl}, a''_{hl}, p''_{hl}, q''_{hl} \in C$.

If both $p'_{12} \neq 0$ and $p''_{12} \neq 0$ then, by (2), $0 = q'_{12} = q_{12} + q_{22} - q_{11}$ and $0 = q''_{12} = q_{12} - q_{22} + q_{11}$. By comparing these last relations and since $char(R) \neq 2$, it follows the contradiction $q_{12} = 0$. Therefore at least one of p'_{12} and p''_{12} must be zero, for instance let $p'_{12} = 0$. Moreover we also notice that $q'_{21} = q_{21} = 0$. Hence we may apply the relation in (7):

$$a'_{22} - a'_{11} - q'_{11}p'_{22} + q'_{22}p'_{11} = 0\,. \tag{8}$$

By computations and using (7) and (6) in (8) we get

$$q_{22}p_{21} - a_{21} = 0\,. \tag{9}$$

Moreover, comparing (9) with (6), and since $p_{21} \neq 0$, we have proved that: $q_{11} = q_{22}$.

Let now $\mu(x) = (1 + e_{21})x(1 - e_{21}) \in Aut(M_2(C))$ and $\nu(x) = (1 - e_{21})x(1 + e_{21}) \in Aut(M_2(C))$. As above, $\mu(a)$, $\mu(q)$, $\mu(p)$, $\nu(a)$, $\nu(q)$, $\nu(p)$ satisfy the same conditions of a, q, p. Denote $\mu(a) = \sum_{hl} a'''_{hl}e_{hl}$, $\nu(a) = \sum_{hl} a^{iv}_{hl}e_{hl}$, $\mu(q) = \sum_{hl} q'''_{hl}e_{hl}$, $\nu(q) = \sum_{hl} q^{iv}_{hl}e_{hl}$, $\mu(p) = \sum_{hl} p'''_{hl}e_{hl}$, $\nu(p) = \sum_{hl} p^{iv}_{hl}e_{hl}$,
for $a'''_{hl}, p'''_{hl}, q'''_{hl}, a^{iv}_{hl}, p^{iv}_{hl}, q^{iv}_{hl} \in C$.

Firstly note that $p'''_{12} = 0$ and $p^{iv}_{12} = 0$. If both $q'''_{21} \neq 0$ and $q^{iv}_{21} \neq 0$ then, by (2), $0 = p'''_{21} = p_{21} + p_{11} - p_{22}$ and $0 = p^{iv}_{12} = p_{21} - p_{11} + p_{22}$. By comparing these last relations and since $char(R) \neq 2$, it follows the contradiction $p_{21} = 0$. Therefore at least one of q'''_{21} and q^{iv}_{21} must be zero, for instance let $q'''_{21} = 0$. By simple computation, since $q_{21} = 0$ and $q_{11} = q_{22}$, it follows the contradiction $0 = q'''_{21} = -q_{12} \neq 0$.

All the previous contradictions show that at least one of q and p must be a diagonal matrix.

In the second step of our proof, we prove that if q is not a diagonal matrix, then $p \in Z(R)$. To do this we recall that, if q is not diagonal, then, by the previous argument, $p_{12} = p_{21} = 0$. Consider here the same above automorphisms $\varphi(x) = (1 + e_{12})x(1 - e_{12}) \in Aut(M_2(C))$ and $\chi(x) = (1 - e_{12})x(1 + e_{12}) \in Aut(M_2(C))$.

If we assume that both $\varphi(p)$ and $\chi(p)$ are not diagonal matrices, then $\varphi(q)$ and $\chi(q)$ are both diagonal. In particular, $0 = q'_{21} = q_{21}$ and

$$0 = q'_{12} = q_{12} + q_{22} - q_{11}$$

and

$$0 = q''_{12} = q_{12} - q_{22} + q_{11}$$

implying $q_{12} = 0$, which contradicts the assumption that q is not diagonal.

On the other hand, if $\varphi(p)$ is a diagonal matrix, then $0 = p'_{12} = p_{11} - p_{22}$, that is p is a central matrix, as required. Analogously p is central, if $\chi(p)$ is diagonal.

In the final step we now prove that, if q is a diagonal matrix, then either $q \in Z(R)$, or $p \in Z(R)$: we introduce again the automorphism φ and notice that, if $\varphi(q)$ is not diagonal, then $\varphi(p)$ must be central, so that $p \in Z(R)$, and we are done.

On the other hand, if $\varphi(q)$ is a diagonal matrix, then $0 = q'_{12} = q_{11} - q_{22}$, that is q is a central matrix, as required. □

Lemma 2.3. *Let $R = M_m(C)$, $m \geq 2$. Suppose that F is an additive map of R defined as follows:*

$$F(x) = ax + qxp$$

for all $x \in R$ and suitable fixed $a, q, p \in Q_r$. If

$$\Big[[F(u), u], F(u)\Big] = 0$$

for all $u \in [R, R]$, then one of the following holds:

1. *$a, q, p \in Z(R)$*
2. *$p \in Z(R)$ and $a + qp \in Z(R)$, that is $F(x) = \lambda x$, for any $x \in R$, with $\lambda = a + qp$;*
3. *$R \subseteq M_2(C)$, $q \in Z(R)$ and $a - qp \in Z(R)$, that is $F(x) = ax + xa + \lambda x$, for any $x \in R$, with $\lambda \in Z(R)$.*

Proof. By Lemmas 2.1 and 2.2, it follows that either $p \in Z(R)$ or $q \in Z(R)$. In any case, F is a generalized derivation of R and the conclusion follows from Theorem 1.2. □

Lemma 2.4. *Let $R = M_m(C)$, $m \geq 2$. Suppose that F is an additive map of R defined as follows:*

$$F(x) = ax + qxp$$

for all $x \in R$ and suitable fixed $a, q, p \in Q_r$. If

$$\Big[[F(u), u], F(u)\Big] = 0$$

for all $u \in R$, then $[F(x), x] = 0$, for all $x \in R$. More precisely, either $a, q, p \in Z(R)$ or $p \in Z(R)$ and $a + qp \in Z(R)$. In any case there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$, for any $x \in R$.

Proof. By Lemmas 2.1 and 2.2, it follows that either $p \in Z(R)$ or $q \in Z(R)$. In any case, $F(x)$ is a generalized derivation of R and the conclusion follows from Theorem 1.1. □

Remark 2.2. If B is a basis of U over C then any element of $T = U *_C C\{x_1, \dots, x_n\}$, the free product over C of the C-algebra U and the free C-algebra $C\{x_1, \dots, x_n\}$, can be written in the form $g = \sum_i \alpha_i m_i$. In this decomposition the coefficients α_i are in C and the elements m_i are B-monomials, that is $m_i = q_0 y_1 q_1 \cdots y_h q_h$, with $q_i \in B$ and $y_i \in \{x_1, \dots, x_n\}$. In [10] it is shown that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if all α_i are zero. Let $a_1, \dots, a_k \in U$ be linearly independent over C and $a_1 g_1(x_1, \dots, x_n) + \dots + a_k g_k(x_1, \dots, x_n) = 0 \in T$, for some $g_1, \dots, g_k \in T$. If, for any i, $g_i(x_1, \dots, x_n) = \sum_{j=1}^{n} x_j h_j(x_1, \dots, x_n)$ and

$h_j(x_1, \ldots, x_n) \in T$, then $g_1(x_1, \ldots, x_n)$,..., $g_k(x_1, \ldots, x_n)$ are the zero element of T. The same conclusion holds if $g_1(x_1, \ldots, x_n)a_1 + \ldots + g_k(x_1, \ldots, x_n)a_k = 0 \in T$, and $g_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} h_j(x_1, \ldots, x_n)x_j$ for some $h_j(x_1, \ldots, x_n) \in T$. (We refer the reader to [1] and [10] for more details on generalized polynomial identities).

Lemma 2.5. *Assume that $q \notin C$ and $p \notin C$. Then $\Psi(x_1, x_2)$ is a non-trivial generalized polynomial identity for R.*

Proof. On the contrary we assume that $\Psi(x_1, x_2)$ is a trivial generalized polynomial identity for R. In light of Remark 2.2 we have that $\Psi(x_1, x_2) = 0 \in T$. We prove that a number of contradictions follows.

Firstly we suppose that $\{1, a, q\}$ is linearly C-dependent, then there exist $\lambda, \mu, \nu \in C$ such that $\lambda a + \mu q + \nu = 0$. Moreover, since $q \notin C$, we also have $\lambda \neq 0$ and $a \neq 0$. Thus $a = \alpha q + \beta$, for suitable $\alpha, \beta \in C$.

In case $\alpha = 0$, then $a = \beta \in C$ and (1) reduces to

$$a\Big[q[x_1, x_2]p, [x_1, x_2]\Big]_2 - \Big[[x_1, x_2], q[x_1, x_2]p\Big]_2 = 0 \in T\,.$$

that is

$$q\Big(\beta[x_1, x_2]p[x_1, x_2]^2 - [x_1, x_2]pq[x_1, x_2]p[x_1, x_2] + 2[x_1, x_2]p[x_1, x_2]q[x_1, x_2]p\Big) + [x_1, x_2]\Big(\beta[x_1, x_2]q[x_1, x_2]p - 2\beta q[x_1, x_2]p[x_1, x_2] - q[x_1, x_2]pq[x_1, x_2]p\Big) = 0 \in T\,. \tag{10}$$

Since $\{1, q\}$ is linearly C-independent, then, by (10) it follows

$$\beta[x_1, x_2]p[x_1, x_2]^2 - [x_1, x_2]pq[x_1, x_2]p[x_1, x_2] + 2[x_1, x_2]p[x_1, x_2]q[x_1, x_2]p = 0 \in T$$

which implies, since also $\{1, p\}$ is linearly C-independent, $2[x_1, x_2]p[x_1, x_2]q[x_1, x_2] \cdot p = 0 \in T$, a contradiction.

Let now $\alpha \neq 0$, then we may write (1) as follows

$$\Big[[q[x_1, x_2](\alpha + p), [x_1, x_2]], q[x_1, x_2](\alpha + p) + \beta[x_1, x_2]\Big] = 0 \in T$$

that is

$$\begin{aligned} &q\Big(\beta[x_1, x_2](\alpha + p)[x_1, x_2]^2 - \\ &[x_1, x_2](\alpha + p)q[x_1, x_2](\alpha + p)[x_1, x_2] + 2[x_1, x_2](\alpha + p)[x_1, x_2]q[x_1, x_2](\alpha + p)\Big) + \\ &[x_1, x_2]\Big(\beta[x_1, x_2]q[x_1, x_2](\alpha + p) - \\ &2\beta q[x_1, x_2](\alpha + p)[x_1, x_2] - q[x_1, x_2](\alpha + p)q[x_1, x_2](\alpha + p)\Big) = 0 \in T\,. \end{aligned} \tag{11}$$

As above, since $\{1, q\}$ is linearly C-independent, then, by (11) it follows

$$\begin{aligned}&\beta[x_1, x_2](\alpha+p)[x_1, x_2]^2-\\&[x_1, x_2](\alpha+p)q[x_1, x_2](\alpha+p)[x_1, x_2]+\\&2[x_1, x_2](\alpha+p)[x_1, x_2]q[x_1, x_2](\alpha+p)=0 \in T\,.\end{aligned} \tag{12}$$

Again since $\{1, p\}$ is linearly C-independent, from relation (12) we get $2[x_1, x_2](\alpha+p)[x_1, x_2]q[x_1, x_2] = 0 \in T$, a contradiction.

We finally consider the case $\{1, a, q\}$ is linearly C-independent and write (1) as follows:

$$\begin{aligned}&a\Big(2[x_1, x_2]^2a[x_1, x_2]+2[x_1, x_2]^2q[x_1, x_2]p-\\&[x_1, x_2]a[x_1, x_2]^2-[x_1, x_2]q[x_1, x_2]p[x_1, x_2]\Big)+\\&q\Big(2[x_1, x_2]p[x_1, x_2]a[x_1, x_2]+\\&2[x_1, x_2]p[x_1, x_2]q[x_1, x_2]p-[x_1, x_2]pa[x_1, x_2]^2-[x_1, x_2]pq[x_1, x_2]p[x_1, x_2]\Big)+\\&[x_1, x_2]\Big(-a[x_1, x_2]a[x_1, x_2]-a[x_1, x_2]q[x_1, x_2]p-\\&q[x_1, x_2]pa[x_1, x_2]-q[x_1, x_2]pq[x_1, x_2]p\Big)=0 \in T\,.\end{aligned} \tag{13}$$

Since $\{1, a, q\}$ is linearly C-independent and by (13) we have

$$\begin{aligned}&2[x_1, x_2]^2a[x_1, x_2]+2[x_1, x_2]^2q[x_1, x_2]p-\\&[x_1, x_2]a[x_1, x_2]^2-[x_1, x_2]q[x_1, x_2]p[x_1, x_2]=0 \in T\,.\end{aligned}$$

Moreover, since $\{1, p\}$ is linearly C-independent, it follows the contradiction

$$2[x_1, x_2]^2q[x_1, x_2]=0 \in T\,. \qquad \square$$

Remark 2.3. *We are now ready to prove Proposition 2.1 and we assume that q is an invertible element of Q_r.*

Proof. The generalized polynomial $\Psi(x_1, x_2)$ is a generalized polynomial identity for R. By [10] it follows that $\Psi(x_1, x_2)$ is a generalized polynomial identity for Q_r.

If either $q \in C$ or $p \in C$, then F is a generalized derivation of R and the conclusion follows from Theorem 1.2.

Therefore we assume that both $q \notin C$ and $p \notin C$. Then, by Lemma 2.5, $\Psi(x_1, x_2)$ is a non-trivial generalized polynomial identity for Q_r. In this case we prove that a contradiction follows.

In view of [20, Theorem 2.5 and Theorem 3.5], we know that both Q_r and $Q_r \bigotimes_C \overline{C}$ are centrally closed, where $\overline{C}$ is the algebraic closure of C. We may replace Q_r by itself or $Q_r \bigotimes_C \overline{C}$ according as C is finite or infinite. Therefore we may assume that Q_r is

centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [26], Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In light of Jacobson's theorem [22, page 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C. Assume first that $dim_C V = k \geq 2$ is a finite positive integer, then $Q_r \cong M_k(C)$ and, by Lemma 2.3, it follows either $p \in C$ or $q \in C$, in any case a contradiction.

Let now $dim_C V = \infty$. As in Lemma 2 in [30], the set $\{[r_1, r_2] | r_i \in Q_r\}$ is dense on Q_r. By the fact that $\Psi(x_1, x_2) = 0$ is a generalized polynomial identity of Q_r, we know that Q_r satisfies

$$\left[ax^2 + qxpx - xax - xqxp, ax + qxp\right] . \tag{14}$$

Let $y_0 \in H$. Since also $[ay_0 + qy_0p, y_0] \in H$, by Litoff's theorem (see Theorem 4.3.11, page 149 in [2]), there exists an idempotent element $e \in H$ such that

$$y_0 , \quad [ay_0 + qy_0p, y_0] \in eQ_re \cong M_k(C)$$

for some positive integer k. Define $f_e : eQ_re \to eQ_re$ by

$$f_e(x) = (ea)x + (eq)x(pe) , \quad \forall x \in eQ_re$$

and notice that both

$$[f_e(x), x] = [eax + eqxpe, x] = e[ax + qxp, x]e , \quad \forall x \in eQ_re$$

and

$$\left[[f_e(x), x], f_e(x)\right] = e\left[[ax + qxp, x], ax + qxp\right]e , \quad \forall x \in eQ_re . \tag{15}$$

Therefore, by (14) and (15), it follows that

$$\left[[f_e(s), s], f_e(s)\right] = 0 , \quad \forall s \in eQ_re . \tag{16}$$

Application of Lemma 2.4 to relation (16) implies that

$$[f_e(s), s] = 0 , \quad \forall s \in eQ_re . \tag{17}$$

In particular, by (17) and the fact that both $y_0 \in eQ_re$ and $[ay_0 + qy_0p, y_0] \in eQ_re$, it follows that

$$0 = [f_e(y_0), y_0] = e[ay_0 + qy_0p, y_0]e = [ay_0 + qy_0p, y_0] . \tag{18}$$

By repeating the same above argument for any element of H, we have that H satisfies the generalized polynomial identity $[ax + qxp, x]$.

Since H is an ideal of Q_r and by Fact 1.2, $[ax + qxp, x]$ is a generalized polynomial identity for Q_r. Since $p \notin C$, there exists $0 \neq v \in V$, such that $\{v, pv\}$ are linearly C-independent. Moreover, by $dim_C V = \infty$ we can fix $w \in V$ such that $\{v, pv, w\}$ are linearly C-independent.

By the density of R, there exist $s \in Q_r$ such that

$$sv = 0 ; \quad s(pv) = q^{-1}w ; \quad sw = -w .$$

Hence it follows the contradiction $0 = [as + qsp, s]v = w \neq 0$. □

3 The proof of Theorem 1.3

As mentioned in the Introduction, we can write $F(x) = ax + d(x)$, for all $x \in R$, where $a \in Q_r$ and d is a skew derivation of R. Let α be the automorphism associated with d. That is $d(xy) = d(x)y + \alpha(x)d(y)$, for all $x, y \in R$.

Remark 3.1. *In light of Theorem 1.2, we get the required conclusion, if one of the following cases occurs:*
- $d = 0$;
- *α is the identity map on R.*

Moreover, Proposition 2.1 proves our result in case simultaneously α is an inner automorphism of R and d is an inner skew derivation of R.
Thus in all that follows, we assume that
- $d \neq 0$;
- *α is not the identity map on R;*
- *either d is not an inner skew derivation, or α is not an inner automorphism of R.*

Remark 3.2. *Since L is a non-central Lie ideal and $char(R) \neq 2$, by [21, p. 4-5] we have that there exists a non-central two-sided ideal I of R such that $[I, R] \subseteq L$.*

In light of previous Remark, there exists a non-central ideal I of R which satisfies

$$\Big[[a[x_1, x_2] + d([x_1, x_2]), [x_1, x_2]], a[x_1, x_2] + d([x_1, x_2])\Big] \tag{1}$$

By using Fact 1.2 it follows that (1) is satisfied by R and Q_r. Then Q_r satisfies

$$\begin{aligned}\Big[&[a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1), [x_1, x_2]],\\ &a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)\Big].\end{aligned} \tag{2}$$

3.1 Let *d* be an inner skew derivation

In this case there exists an element $c \in Q_r$ such that $d(x) = cx - \alpha(x)c$, for all $x \in R$. If $c = 0$, then F is a generalized derivation of R and the conclusion follows from Theorem 1.2. Thus we assume $c \neq 0$.
By (1), Q_r satisfies

$$\Big[[(a + c)[x_1, x_2] - \alpha([x_1, x_2])c, [x_1, x_2]], (a + c)[x_1, x_2] - \alpha([x_1, x_2])c\Big]. \tag{3}$$

Moreover, by Remark 3.1 we may assume α is not an inner automorphism of Q_r. Notice that the $\alpha(x_i)$-word degree in (3) is 2. Hence, since α is outer and either $char(R) = 0$ or $char(R) \geq 3$, then, by [12, Theorem 3] and by (3), we have that Q_r satisfies

$$\Big[[(a + c)[x_1, x_2] + [x_3, x_4]c, [x_1, x_2]], (a + c)[x_1, x_2] - [x_3, x_4]c\Big]. \tag{4}$$

In particular Q_r satisfies

$$\Big[[(a+c)[x_1,x_2],[x_1,x_2]],(a+c)[x_1,x_2]\Big] \tag{5}$$

and by Theorem 1.2, it follows $a+c=\lambda\in C$. Thus (4) reduces to

$$\Big[[[x_3,x_4]c,[x_1,x_2]],\lambda[x_1,x_2]-[x_3,x_4]c\Big]. \tag{6}$$

For $[x_3,x_4]=[x_1,x_2]$ in (6), it follows that

$$\Big[[[x_1,x_2](-a),[x_1,x_2]],[x_1,x_2]a\Big] \tag{7}$$

is a generalized identity for Q_r, and, by using again the result in Theorem 1.2, we get $a\in C$, that is also $0\neq c\in C$. Therefore, by (6) and easy computations, one has that Q_r satisfies the polynomial identity

$$\lambda\Big[[x_3,x_4],[x_1,x_2]\Big]_2-c\Big[[x_1,x_2],[x_3,x_4]\Big]_2. \tag{8}$$

By the well known Posner's theorem, there exists a suitable field K such that Q_r and $M_t(K)$, the ring of $t\times t$ matrices over K, satisfy the same polynomial identities. In particular $M_t(K)$ satisfies (8) and $k\geq 2$, since Q_r is not commutative. On the other hand, for $[x_1,x_2]=e_{21}$ and $[x_3,x_4]=e_{12}$ in (8), it follows the contradiction $0=2\lambda e_{21}+2ce_{12}\neq 0$.

3.2 Let *d* be an outer skew derivation

Starting from (4), since d is outer and by Fact 1.1, we have that Q_r satisfies

$$\begin{aligned}\Big[[a[x_1,x_2]+tx_2+\alpha(x_1)z-zx_1-\alpha(x_2)t,[x_1,x_2]],\\ a[x_1,x_2]+tx_2+\alpha(x_1)z-zx_1-\alpha(x_2)t\Big].\end{aligned} \tag{9}$$

For $t=z=0$ in (9) it follows that Q_r satisfies

$$\Big[[a[x_1,x_2],[x_1,x_2]],a[x_1,x_2]\Big]$$

By Theorem 1.2, we get $a\in C$ and (9) reduces to

$$\begin{aligned}\Big[[tx_2+\alpha(x_1)z-zx_1-\alpha(x_2)t,[x_1,x_2]],\\ a[x_1,x_2]+tx_2+\alpha(x_1)z-zx_1-\alpha(x_2)t\Big].\end{aligned} \tag{10}$$

In particular Q_r satisfies the blended component

$$\Big[[tx_2-\alpha(x_2)t,[x_1,x_2]],a[x_1,x_2]+tx_2-\alpha(x_2)t\Big]. \tag{11}$$

Assume that α is an outer automorphism of Q_r. As above, we notice that the $\alpha(x_i)$-word degree in (11) is 2. Hence, since α is outer and either $char(R) = 0$ or $char(R) \geq 3$, then, by [12, Theorem 3] and by (11), we have that

$$\Big[[tx_2 - vt, [x_1, x_2]], a[x_1, x_2] + tx_2 - vt\Big] \tag{12}$$

is a polynomial identity for Q_r. Thus there exists a suitable field K such that Q_r and $M_n(K)$ satisfies the same polynomial identities, in particular $M_n(K)$ satisfies (12) and $n \geq 2$, since Q_r is not commutative. On the other hand, for $v = 0$, $[x_1, x_2] = [e_{22}, e_{21}] = e_{21}$ and $t = e_{12}$ it follows the contradiction $0 = e_{21} \neq 0$.

Assume now there exists an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$, for all $x \in Q_r$. Moreover $q \notin C$, since α is not the identity map. By (11), Q_r satisfies

$$\Big[[tx_2 - qx_2q^{-1}t, [x_1, x_2]], a[x_1, x_2] + tx_2 - qx_2q^{-1}t\Big]$$

and replacing t by qt, it follows that

$$a\Big[q[t, x_2], [x_1, x_2]\Big]_2 - \Big[[x_1, x_2], q[t, x_2]\Big]_2 \tag{13}$$

is satisfied by Q_r. Since $q \notin C$, then (13) is a non-trivial generalized polynomial identity for Q_r. By [26], Q_r is a primitive ring having nonzero socle with the field C as its associated division ring. By [22] (p. 75) Q_r is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing nonzero linear transformations of finite rank.

Assume first that $dim_C V = k \geq 3$. Since $q \notin C$, then there exists $v \in V$ such that v, qv are linearly C-independent, moreover, since $dim_C V \geq 3$, it follows that there exists $w \in V$ such that v, qv, w are linearly C-independent. By the density of Q_r, there are $x, y, t \in Q_r$ such that

$$xv = v\,, \quad yv = 0\,, \quad tv = -w$$

$$xw = v\,, \quad yw = v\,, \quad tw = v$$

$$xqv = 0\,, \quad yqv = w\,, \quad tqv = v\,.$$

Therefore, by (13), we have the following contradiction:

$$0 = \left(a\Big[q[t, x_2], [x_1, x_2]\Big]_2 - \Big[[x_1, x_2], q[t, x_2]\Big]_2\right)v = -v - 2qv \neq 0\,.$$

On the other hand, if $dim_C V = 2$, then $Q_r \cong M_2(C)$. and we write $q = \sum q_{ij}e_{ij}$, with $q_{ij} \in C$. Let $[x, y] = [-e_{12}, e_{11}] = e_{12}$ and $[t, y] = [e_{21}, e_{11}] = e_{21}$ in the relation (13). Thus it follows that

$$-2ae_{12}qe_{22} - qe_{21}qe_{22} - e_{12}qe_{21}qe_{21} + 2qe_{22}qe_{21} = 0\,. \tag{14}$$

The $(2, 1)$-entry of the matrix (14) is $2q_{22}^2 = 0$, which implies $q_{22} = 0$. In light of this, the $(1, 2)$-entry of (14) reduces to $-q_{12}^2 = 0$, that is $q_{12} = 0$. Therefore the matrix q should have the second column zero, wich contradicts the fact that q is invertible.

We conclude our paper with the following consequence of Theorem 1.3:

Theorem 3.1. *Let R be a prime ring of characteristic different from 2, F a non-zero generalized skew derivation of R. If*

$$\Big[[F(x), x], F(x)\Big] = 0$$

for all $x \in R$, then R is commutative.

Proof. Firstly we notice that, under our assumption and in light of Theorem 1.3, F must be a generalized derivation of R. Therefore, by applying Theorem 1.1 we get the required conclusion. □

We also have:

Theorem 3.2. *Let R be a prime ring of characteristic different from 2, F a non-zero skew derivation of R and L a Lie ideal of R. If*

$$\Big[[F(u), u], F(u)\Big] = 0$$

for all $u \in L$, then $L \subseteq Z(R)$.

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Basudeb Dhara

Generalized Skew-Derivations on Lie Ideals in Prime Rings

Abstract: Let R be a prime ring with characteristic different from 2, Q_r be its right Martindale quotient ring, C be its extended centroid, F a nonzero generalized skew derivation of R, L a noncentral Lie ideal of R, $n \geq 1$, $m \geq 0$ be fixed integers and $0 \neq a \in R$. If $au^m F(u)^n = 0$ for all $u \in L$, then one of the following holds: (i) $m = 0$ and there exists $b \in Q_r$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$; (ii) R satisfies s_4.

Keywords: Prime ring; skew derivation; generalized skew derivation; extended centroid.

1 Introduction

Let R be an associative prime ring with center $Z(R)$. The commutator of x and y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$ for $x, y \in R$. An additive mapping $d\colon R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F\colon R \to R$ is called a generalized derivation if there exists a derivation $d\colon R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Evidently, any derivation is a generalized derivation. The standard polynomial identity s_4 in four variables is defined as $s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$ where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or odd permutation in symmetric group S_4.

In [16], Sharma and Dhara proved that if R is a prime ring with a nonzero derivation d, L be its nonzero Lie ideal and $a \in R$ such that $au^n d(u)^m = 0$ for all $u \in L$, where $n \geq 1$ and $m \geq 1$ are fixed integers, then one of the following holds:
(1) $a = 0$ or $d(L) = 0$ if char $R \neq 2$;
(2) $a = 0$ or $d(R) = 0$ if $[L, L] \neq 0$ and $R \neq M_2(F)$, the 2×2 matrices over the field F of two elements.

For noncommutative Lie ideal L of R, Dhara and Sharma obtained results [12] that if $a \in R$ such that $au^s d(u)^n u^t = 0$ for all $u \in L$, where $s(\geq 0), t(\geq 0), n(\geq 1)$ are fixed integers, then either $a = 0$ or $d(R) = 0$ unless char $R = 2$ and R satisfies s_4, the standard identity in four variables.

In [11], Dhara and De Filippis studied a situation for generalized derivation. They proved for prime ring R that if H is a generalized derivation of R and L a noncommutative Lie ideal of R such that $u^s H(u) u^t = 0$ for all $u \in L$, where $s \geq 0, t \geq 0$ are fixed

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integers, then $H(x) = 0$ for all $x \in R$ unless char $R = 2$ and R satisfies s_4, the standard identity in four variables.

Recently, Du and Wang [13] proved a result for generalized derivations as follows: *Let R be a prime ring, U be its Utumi ring of quotients, H a nonzero generalized derivation of R, L a noncentral Lie ideal of R and* $0 \neq a \in R$. *Suppose that* $au^s H(u)^n u^t = 0$ *for all* $u \in L$, *where* $s, t \geq 0$ *and* $n \geq 1$ *are fixed integers. Then one of the following holds:*
(1) $s = 0$ *and there exists* $b \in U$ *such that* $H(x) = bx$ *for all* $x \in R$ *with* $ab = 0$;
(2) R satisfies s_4.
In the present paper, we shall study the result by considering H as a generalized skew derivation.

We recall now the definition of generalized skew derivations of R. Let R be an associative ring and α be an automorphism of R. An additive mapping $d\colon R \to R$ is called a skew derivation of R if $d(xy) = d(x)y+\alpha(x)d(y)$ holds for all $x, y \in R$, where α is an automorphism of R. An additive mapping $F\colon R \to R$ is said to be a generalized skew derivation of R, if there exists a skew derivation d of R with associated automorphism α such that $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$, d is said to be an associated skew derivation of F and α is called an associated automorphism of F. The mapping $F(x) = ax + \alpha(x)b$ for some $a, b \in R$ and $\alpha \in Aut(R)$, is an example of generalized skew derivation, which is called as inner generalized skew derivation. In particular, if $a = -b$, then G is called inner skew derivation of R. If a generalized skew derivation (respectively, a skew derivation) is not inner, then it is usually called outer. Thus the concept of generalized skew derivation generalized the notions of skew derivation and generalized derivation.

Recently, Ashraf et al. [1] proved a result for generalized skew derivation as follows: *Let R be a prime ring of characteristic different from* 2, *C its extended centroid, L a noncentral Lie ideal of R and* $m, n, t \geq 1$ *fixed integers. Suppose that F is a nonzero generalizes skew derivation of R such that* $u^m F(u)^t u^n = 0$ *for all* $u \in L$. *Then* $dim_C RC = 4$.

In the present paper our motivation is to study the above situation with left annihilator condition, when $n = 0$. More precisely, we prove the following theorem.

Theorem 1.1. *Let R be a prime ring with characteristic different from* 2, Q_r *be its right Martindale quotient ring, C be its extended centroid, F a nonzero generalized skew derivation of R, L a noncentral Lie ideal of R,* $n \geq 1$, $m \geq 0$ *be fixed integers and* $0 \neq a \in R$. *If* $au^m F(u)^n = 0$ *for all* $u \in L$, *then one of the following holds:*
(i) $m = 0$ *and there exists* $b \in Q_r$ *such that* $F(x) = bx$ *for all* $x \in R$ *with* $ab = 0$;
(ii) R satisfies s_4.

Open Question: *Let R be a prime ring with characteristic different from* 2, Q_r *be its right Martindale quotient ring, C be its extended centroid, F a nonzero generalized skew derivation of R, L a noncentral Lie ideal of R,* $n \geq 1$, $m \geq 0$, $t \geq 0$ *be fixed integers and* $0 \neq a \in R$. *If* $au^m F(u)^n u^t = 0$ *for all* $u \in L$, *then what will be the form of F or structure of R?*

2 Preliminaries

Let R be a prime ring and Q_r be the right Martindale quotient ring of R. Then $C = Z(Q_r)$ is called the extended centroid of R which is a field. It should be remarked that Q_r is a centrally closed prime C-algebra. We refer the reader to [2] for the definitions and the related properties of these objects.

We need the following facts.

Fact 2.1. *It is well known that any automorphisms, derivations and skew derivations of R can be uniquely extended in Q_r. In [5], Chang extended the definition of generalized skew derivation to the right Martindale quotient ring Q_r of R as follows: a (right) generalized skew derivation $F\colon Q_r \to Q_r$ is an additive mapping such that $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in Q_r$, where d is a skew derivation of R and α is an automorphism of R. In particular, for $x = 1$, we have $F(y) = F(1)y + d(y) = ay + d(y)$ for all $y \in Q_r$, where $a = F(1) \in Q_r$.*

Fact 2.2. *Chuang and Lee [10] investigated polynomial identities with skew derivations. They prove that if $F(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D is an outer skew derivation of R, then R also satisfies the generalized polynomial identity $F(x_i, y_i)$, where x_i and y_i are distinct indeterminates. Furthermore, they observe [10, Theorem 1] that in the case $F(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, D is an outer skew derivation of R and α is an outer automorphism of R, then R also satisfies the generalized polynomial identity $F(x_i, y_i, z_i)$, where x_i, y_i and z_i are distinct indeterminates.*

Fact 2.3. *Let d be a non-zero skew derivation of R. If $\Phi(x_1, \dots, x_n, d(x_1), \dots, d(x_n))$ is a skew-differential polynomial identity of R, then either d is an inner skew derivation of R or R satisfies the generalized polynomial identity $\Phi(x_1, \dots, x_n, y_1, \dots, y_n)$ (see [10]).*

Fact 2.4. *Let R be a prime ring and I be a two-sided ideal of R. Then, I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [6]). Furthermore, I, R, and Q_r satisfy the same generalized polynomial identities with automorphisms (see [8, Theorem 1])*

Fact 2.5. *If R is a prime ring, then the following are equivalent:*
(i) R satisfies s_4;
(ii) R is commutative or R embeds in $M_2(F)$ for F a field;
(iii) R is algebraic of bounded degree 2 over C;
(iv) R satisfies $[[x^2, y], [x, y]]$ (see [4, Lemma 1]).

3 Proof of Main Results

Lemma 3.1. *(See [13, Proof of Theorem 1.1]) Let R be a prime ring, $0 \neq a \in R$ and $b, c \in Q_r$ such that $a[x, y]^m(b[x, y] + [x, y]c)^n = 0$ for all $x, y \in R$, where $m \geq 0$, $n \geq 1$ are fixed integers. (i) If $m = 0$, then $c \in C$ and $a(b+c) = 0$, unless R satisfies s_4; (ii) if $m > 0$, then $c \in C$ and $b + c = 0$, unless R satisfies s_4.*

Proposition 3.1. *Let R be a dense subring of the ring of linear transformations of a vector space V over a division ring D and let R contains nonzero linear transformations of finite rank. Let $m \geq 0$, $n \geq 1$ be fixed integers, α be an automorphism of R and for some $a, b \in R$, $F(x) = ax + \alpha(x)b$ such that $px^mF(x)^n = 0$ for all $x \in [R, R]$. If $F \neq 0$ and R does not satisfy s_4, then one of the following holds*
(i) $m = 0$ and $F(x) = (a + b)x$ for all $x \in R$ with $p(a + b) = 0$;
(ii) $dim_D V \leq 2$.

Proof. By hypothesis, we have

$$p[x, y]^m(a[x, y] + \alpha([x, y])b)^n = 0 \tag{1}$$

for all $x, y \in R$. By [7], R is a GPI-ring. Moreover, R is a primitive ring having nonzero socle with associated division ring D. By [14, pp. 79], there exists a semi-linear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in R$. Hence

$$p[x, y]^m(a[x, y] + T[x, y]T^{-1}b)^n = 0 \tag{2}$$

for all $x, y \in R$.

If for any $v \in V$, v and $T^{-1}bv$ are linearly D-dependent, then by standard argument there exists a $\lambda \in D$ such that $T^{-1}bv = \lambda v$ for all $v \in V$. In this case

$$(ax + \alpha(x)b)v = (ax + TxT^{-1}b)v = axv + T(\lambda xv) = axv + bxv = (a + b)xv\,.$$

Thus for all $v \in V$, $(ax+\alpha(x)b-(a+b)x)v = 0$, that is $(ax+\alpha(x)b-(a+b)x)V = 0$. Since V is faithful, $ax+\alpha(x)b-(a+b)x = 0$ for all $x \in R$. By hypothesis, $p[x, y]^m((a+b)[x, y])^n = 0$ for all $x, y \in R$. Then by Lemma 3.1, either $a + b = 0$ or $m = 0$ and $p(a + b) = 0$, unless R satisfies s_4. If $a + b = 0$, then $F(x) = ax + \alpha(x)b = (a + b)x = 0$ for all $x \in R$, a contradiction. If $m = 0$ and $p(a + b) = 0$, then $F(x) = ax + \alpha(x)b = (a + b)x$ for all $x \in R$, which is our conclusion (1).

Next we assume that there exists some $v \in V$ such that v and $T^{-1}bv$ are linearly D-independent.

Let $\dim_D V \geq 3$. Then there exists $w \in V$ such that v, $T^{-1}bv$, w are linearly D-independent. By density of R, there exist $x, y \in R$, such that

$$xv = w,\quad xT^{-1}bv = 0,\quad xw = T^{-1}v - T^{-1}av\,;$$
$$yv = 0,\quad yT^{-1}bv = w,\quad yw = -v\,.$$

Then $0 = p[x, y]^m(a[x, y] + T[x, y]T^{-1}b)^nv = pv$.

This implies that if $pv \neq 0$, by contradiction, we can say that v and $T^{-1}bv$ are linearly D-dependent. Now choose $v \in V$ such that v and $T^{-1}bv$ are linearly D-independent.

Then $pv = 0$. Set $W = Span_D\{v, T^{-1}bv\}$. Since $p \neq 0$, there exists $w \in V$ such that $pw \neq 0$ and then $p(v-w) = pw \neq 0$. By the previous argument we have that w, $T^{-1}bw$ are linearly D-dependent and $(v-w)$, $T^{-1}b(v-w)$ too. Thus there exist $\alpha, \beta \in D$ such that $T^{-1}bw = \alpha w$ and $T^{-1}b(v-w) = \beta(v-w)$. Then $T^{-1}bv = \beta(v-w) + T^{-1}bw = \beta(v-w) + \alpha w$ i.e., $(\alpha - \beta)w = T^{-1}bv - \beta v \in W$. Now $\alpha = \beta$ implies that $T^{-1}bv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$. Again, if $u \in V$ with $pu = 0$ then $p(w+u) \neq 0$. So, $w+u \in W$ forcing $u \in W$. Thus it is observed that $w \in V$ with $pw \neq 0$ implies $w \in W$ and $u \in V$ with $pu = 0$ implies $u \in W$. This implies that $V = W$ i.e., $\dim_D V = 2$, a contradiction. Thus we have $\dim_D V \leq 2$. □

Proposition 3.2. *Let R be a noncommutative prime ring of characteristic different from 2, let Q_r be its right Martindale ring of quotients, and C be its extended centroid. Let I be a two-sided ideal of R, $m \geq 0$, $n \geq 1$ fixed integers, α be an automorphism of R which is not identity map and for some $a, b \in R$, $F(x) = ax + \alpha(x)b$ such that $px^m F(x)^n = 0$ for all $x \in [I, I]$. If $F \neq 0$, then one of the following holds:*
(i) $m = 0$ and $F(x) = ax$ for all $x \in R$ with $pa = 0$;
(ii) R satisfies s_4.

Proof. By hypothesis, I satisfies

$$p[x,y]^m(a[x,y] + \alpha([x,y])b)^n = 0\,. \tag{3}$$

Assume first that α is an inner automorphism of R, i.e., there exists an invertible element q in Q_r such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. Then I satisfies

$$p[x,y]^m(a[x,y] + q[x,y]q^{-1}b)^n = 0\,. \tag{4}$$

Since I, R and Q_r satisfies the same generalized polynomial identities [6], Q_r satisfies

$$p[x,y]^m(a[x,y] + q[x,y]q^{-1}b)^n = 0\,. \tag{5}$$

If $q^{-1}b \in C$, then $F(x) = (a+b)x$ for all $x \in R$, and hence result follows by Lemma 3.1. So we assume that $q^{-1}b \notin C$. Then (5) becomes a nontrivial GPI for Q_r. By Martindale Theorem [15], Q_r is isomorphic to a dense ring of linear transformations of a vector space V over D, where D is a finite dimensional division algebra over C. Then by Proposition 3.1, either Q_r satisfies s_4 or $dim_D V \leq 2$. In the last case, when $dim_D V \leq 2$, we have $Q_r \cong M_k(D)$, for $k \leq 2$.

If C is finite, then $dim_C D < \infty$ implies that D is also finite. Therefore, $D \cong C$ is a field by Wedderburn's theorem. This implies $Q_r \cong M_k(C)$ for $k \leq 2$. But if $k = 1$, then R is commutative, a contradiction. Thus we must have $Q_r \cong M_2(C)$, that is, R satisfies s_4.

If C is infinite, then let $\overline{C}$ be the algebraic closure of C. We know that Q_r and $Q_r \bigotimes_C \overline{C}$ satisfy the same generalized polynomial identities. Moreover,

$$Q_r \otimes_C \overline{C} \cong M_k(D) \otimes_C \overline{C} \cong M_k(D \otimes_C \overline{C}) \cong M_t(\overline{C})\,,$$

for some $t \geq 1$. Thus since $Q_r \bigotimes_C \overline{C}$ satisfies (5), by Proposition 3.1, we must have $t = 2$, that is $R \subseteq M_2(\overline{C})$. This implies that R satisfies s_4.

Next we assume that α is an outer automorphism of R. Then by [8], Q_r satisfies

$$p[x,y]^m(a[x,y] + \alpha([x,y])b)^n = 0 . \tag{6}$$

Then by [7, Main Theorem], Q_r is a GPI-ring. Thus Q_r is a primitive ring having nonzero socle and its associated division ring D is a finite dimensional over C. Thus by Proposition 3.1, we are to consider the case $dim_D V \leq 2$.

If C is finite, then it follows that D is also finite. By Wedderburn's Theorem D is a field and by Proposition 3.1 we also have $dim_D V = 2$, that is, R satisfies s_4. Hence we assume that C is infinite. If α is not Frobenius, then by main Theorem in [8] Q_r satisfies

$$p[x,y]^m(a[x,y] + [s,t]b)^n = 0 . \tag{7}$$

If R satisfies s_4, then we have our conclusion (2). So we assume that R does not satisfy s_4. In particular, for $s = t = 0$, Q_r satisfies

$$p[x,y]^m(a[x,y])^n = 0 , \tag{8}$$

which implies by Lemma 3.1 that either $m > 0$ and $a = 0$ or $m = 0$ and $pa = 0$. If $m > 0$ and $a = 0$, then by (7) Q_r satisfies

$$p[x,y]^m([s,t]b)^n = 0 , \tag{9}$$

which implies again by Lemma 3.1 that $b = 0$. This implies $F = 0$, a contradiction. On the other hand, if $m = 0$ and $pa = 0$, then (7) yields

$$p(a[x,y] + [s,t]b)^n = 0 . \tag{10}$$

In particular, Q_r satisfies $p([s,t]b)^n = 0$ and $p(a[x,y])^n = 0$. Both cases give by Lemma 3.1 that $b = 0$ and $pa = 0$. Thus we obtain our conclusion (1).

If α is Frobenius, then char $(Q_r) = p > 0$, otherwise $\alpha(\lambda) = \lambda$ for all $\lambda \in C$ and α must be inner by [2, Theorem 4.7.4], a contradiction. Moreover, $\alpha(\lambda) = \lambda^{p^t}$ for all $\lambda \in C$, where t is some fixed integer. Replacing x with λx, where $\lambda \neq 0$ in (23), we have that Q_r satisfies

$$p\lambda^m[x,y]^m(\lambda a[x,y] + \lambda^{p^t}\alpha([x,y])b)^n = 0 \tag{11}$$

that is

$$p\lambda^{m+n}[x,y]^m(a[x,y] + \lambda^{p^t-1}\alpha([x,y])b)^n = 0 . \tag{12}$$

Since $\lambda \neq 0$,

$$p[x,y]^m(a[x,y] + \lambda^{p^t-1}\alpha([x,y])b)^n = 0 . \tag{13}$$

Let $\varphi_1 = a[x,y]$, $\varphi_2 = \alpha([x,y])b$ and $\gamma = \lambda^{p^t-1}$. Then we have from (13) that Q_r satisfies

$$0 = p[x,y]^m(\varphi_1 + \gamma\varphi_2)^n = \sum_{i=0}^{n} \psi_i(\varphi_1, \varphi_2)\gamma^i , \tag{14}$$

where $\psi_i(\varphi_1, \varphi_2)$ denotes the sum of all monomials with φ_1 degree i and φ_2 degree $n-i$, for $i = 0, 1, \dots, n$. In particular, $\psi_0(\varphi_1, \varphi_2) = p[x, y]^m(\varphi_2)^n$ and $\psi_n(\varphi_1, \varphi_2) = p[x, y]^m(\varphi_1)^n$. Then replacing λ with $1, \lambda, \lambda^2, \dots, \lambda^n$, that is γ with $1, \gamma, \gamma^2, \dots, \gamma^n$ in (14), we get the following homogeneous system of equations respectively.

$$\psi_0 + \psi_1 + \psi_2 + \cdots + \psi_n = 0\,,$$
$$\psi_0 + \gamma\psi_1 + \gamma^2\psi_2 + \cdots + \gamma^n\psi_n = 0\,,$$
$$\psi_0 + \gamma^2\psi_1 + \gamma^4\psi_2 + \cdots + \gamma^{2n}\psi_n = 0\,,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$\psi_0 + \gamma^n\psi_1 + \gamma^{2n}\psi_2 + \cdots + \gamma^{n^2}\psi_n = 0\,.$$

Denote γ^j by μ_j and then above system becomes

$$\psi_0 + \psi_1 + \psi_2 + \cdots + \psi_n = 0\,,$$
$$\psi_0 + \mu_1\psi_1 + \mu_2\psi_2 + \cdots + \mu_n\psi_n = 0\,,$$
$$\psi_0 + \mu_1^2\psi_1 + \mu_2^2\psi_2 + \cdots + \mu_n^2\psi_n = 0\,,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$\psi_0 + \mu_1^n\psi_1 + \mu_2^n\psi_2 + \cdots + \mu_n^n\psi_n = 0\,.$$

Moreover, since C is infinite, there exists infinitely many elements $\gamma \in C$ such that $\gamma^i \neq 1$ for $i = 1, \dots, n$, that is there exists infinitely many $\mu \in C$ such that $\mu \neq 1$. Hence, the Vandermonde determinant $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \mu_1 & \cdots & \cdots & \mu_n \\ 1 & \mu_1^2 & \cdots & \cdots & \mu_n^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \mu_1^n & \cdots & \cdots & \mu_n^n \end{vmatrix} = \pm(1-\mu_1) \prod_{1 \le i \le j \le 2} (\mu_i - \mu_j)$ is not zero. Thus we can solve the above system of equations and obtain that $\psi_i = 0$ for $i = 0, 1, \dots, n$. In particular, $\psi_0 = 0$ and $\psi_n = 0$ implies

$$p[x, y]^m(\alpha([x, y])b)^n = 0 \tag{15}$$

and

$$p[x, y]^m(a[x, y])^n = 0\,. \tag{16}$$

By Lemma 3.1, (16) implies (i) $m = 0$ and $pa = 0$ or (ii) $m > 0$ and $a = 0$.

Let us assume first that $m = 0$ and $pa = 0$. Then (15) implies

$$p(\alpha([x, y])b)^n = 0\,. \tag{17}$$

Since $\dim_D V \le 2$, by [9, Lemma 1] Q_r satisfies

$$p(\alpha([x, y])b)^2 = 0\,. \tag{18}$$

Since any $\alpha(x)$ and $\alpha(y)$-word degree are 2 and either char $(R) = 0$ or char $(R) \geq 3$, then, by [8, Theorem 3], Q_r satisfies

$$p([s,t]b)^2 = 0\,, \tag{19}$$

which implies by Lemma 3.1 that $b = 0$, a contradiction. Thus we have our conclusion (1).

Next assume that $m > 0$ and $a = 0$. Then by (6), Q_r satisfies

$$p[x,y]^m(\alpha([x,y])b)^n = 0\,. \tag{20}$$

Since $\dim_D V \leq 2$, by [9, Lemma 1] Q_r satisfies

$$p[x,y]^m(\alpha([x,y])b)^2 = 0\,. \tag{21}$$

Since any $\alpha(x)$ and $\alpha(y)$-word degree are 2 and either char $(R) = 0$ or char $(R) \geq 3$, then, by [8, Theorem 3], Q_r satisfies

$$p[x,y]^m([s,t]b)^2 = 0\,. \tag{22}$$

By Lemma 3.1, $b = 0$. Then F becomes $F(x) = ax$ for all $x \in R$, which implies again by using Lemma 3.1 that $a = 0$. Thus $F = 0$, a contradiction. Hence the proof is completed. □

Proof of Theorem 1.1: The generalized skew derivation F has its form $F(x) = bx + d(x)$ for all $x \in R$, where $a \in Q_r$ and d is a skew-derivation of R. Let α be the associated automorphism to F. If $\alpha = I_{id}$ is an identity map of R or $d = 0$, then F becomes a generalized derivation of R and in this case result follows by [13, Theorem 1.1]. Thus we assume that $\alpha \neq I_{id}$. Since L is a noncentral Lie ideal of R and char $(R) \neq 2$, by [3, Lemma 1] there exists a nonzero ideal I of R such that $[I,I] \subseteq L$. Hence, by our assumption we have $a[x,y]^m(b[x,y] + d([x,y]))^n = 0$ for all $x,y \in I$ and also for all $x,y \in Q_r$ (see [10, Theorem 2]). This implies that Q_r satisfies

$$a[x,y]^m(b[x,y] + d(x)y + \alpha(x)d(y) - d(y)x - \alpha(y)d(x))^n = 0\,. \tag{23}$$

If $d(x) = cx - \alpha(x)c$ is an inner skew derivation of Q_r, then $F(x) = (b+c)x - \alpha(x)c$ for all $x \in Q_r$. Since α is not an identity map, by Proposition 3.2, $m = 0$, $c = 0$ and $ab = 0$. This is our conclusion (1).

Next if d is not inner skew-derivation of R, by [10, Theorem 1] Q_r satisfies

$$a[x,y]^m(b[x,y] + sy + \alpha(x)t - tx - \alpha(y)s)^n = 0\,. \tag{24}$$

Replacing s with $c'x - \alpha(x)c'$ and t with $c'y - \alpha(y)c'$ for some $c' \neq 0$, (24) yields

$$a[x,y]^m(b[x,y] + c'[x,y] - \alpha([x,y])c')^n = 0 \tag{25}$$

for all $x,y \in Q_r$. By Proposition 3.2, it yields $c' = 0$, unless R satisfies s_4, a contradiction. Thus we conclude that R satisfies s_4. This completes the proof of theorem.

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Moulay Abdallah Idrissi, Abdellah Mamouni, and Lahcen Oukhtite

On generalized derivations and commutativity of prime rings with involution

Abstract: In this paper we investigate commutativity in rings with involution admitting a generalized derivation satisfying certain algebraic identities. Some well-known results characterizing commutativity of prime rings have been generalized. Moreover, we provide examples to show that the assumed restrictions cannot be relaxed.

Keywords: Prime ring; involution; commutativity; derivation; generalized derivation.

1 Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. Recall that R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$.

An additive map $*: R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2. An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case it is straightforward to check that $S(R) \cap Z(R) \neq (0)$.

An additive map $d: R \longrightarrow R$ is called a *derivation* on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$ be a fixed element, the map $d: R \longrightarrow R$ defined by $d(x) = [a, x] = ax - xa$ for all $x \in R$ is a derivation on R called the *inner derivation* induced by a. Many results in literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. A well known result due to Posner [14] states that if d is a derivation of a prime ring R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. In [7] Lanski generalizes the result of Posner by considering a derivation d such that $[d(x), x] \in Z(R)$ for all x in a nonzero Lie ideal U of R.

More recently several authors consider similar situation in the case the derivation d is replaced by a generalized derivation. More specifically an additive map $F: R \longrightarrow R$

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is said to be a generalized derivation if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Basic examples of generalized derivations are the usual derivations on R and left R-module mappings from R into itself. An important example is a map of the form $F(x) = ax + xb$ for some elements $a, b \in R$; such generalized derivations are called *inner*. Generalized derivations have been primarily studied on operator algebras thus any investigation from the algebraic point of view might be interesting (see for example [4, 6] and [8]).

During the last two decades, many authors have studied commutativity of prime and semi-prime rings admitting suitably constrained additive mappings acting on appropriate subsets of the rings. Moreover, many of obtained results extend other ones previously proven just for the action of the considered mapping on the whole ring. In this direction, the recent literature contains numerous results on commutativity in prime and semi-prime rings admitting suitably constrained derivations and generalized derivations, and several authors have improved these results by considering rings with involution (for example, see [1, 2, 9, 11] and [12]).

The purpose of the present paper is to continue this line of investigation and study the structure of a prime ring with involution of the second kind admitting a generalized derivation satisfying more specific algebraic identities.

2 Main results

We first fix the following lemma which shall be used frequently throughout the text.

Lemma 2.1 ([10], Fact 1.). *Let $(R, *)$ be a 2-torsion free prime ring with involution provided with a derivation d. Then $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ implies that $d(z) = 0$ for all $z \in Z(R)$.*

Theorem 2.1. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a nonzero derivation d, then the following assertions are equivalent:*
(1) $[F(x), d(x^*)] = \pm[x, x^*]$ for all $x \in R$.
(2) $[F(x), d(x^*)] = 0$ for all $x \in R$.
(3) R is commutative.

Proof. Obviously, (3) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2).
(1) $\Longrightarrow$ (3) For all $x \in R$, we suppose that

$$[F(x), d(x^*)] = [x, x^*] \tag{1}$$

Linearizing (1), we get $[F(x), d(y^*)] + [F(y), d(x^*)] = [x, y^*] + [y, x^*]$ so that

$$[F(x), d(y)] + [F(y^*), d(x^*)] = [x, y] + [y^*, x^*] \quad \text{for all } x, y \in R . \tag{2}$$

Substituting yh for y in (2), where $h \in Z(R) \cap H(R)\backslash\{0\}$ and using (2) we obtain

$$([F(x), y] + [y^*, d(x^*)])d(h) = 0 \quad \text{for all} \ \ x, y \in R .$$

Using the primeness hypothesis, it follows that $[F(x), y] + [y^*, d(x^*)] = 0$ or $d(h) = 0$. Suppose that

$$[F(x), y] + [y^*, d(x^*)] = 0 \quad \text{for all} \ \ x, y \in R . \tag{3}$$

Replacing y by ys in (3), where $s \in Z(R) \cap S(R)\backslash\{0\}$, it's easy to verify that

$$[F(x), y] - [y^*, d(x^*)] = 0 \quad \text{for all} \ \ x, y \in R . \tag{4}$$

Adding relations (3) and (4) we find that

$$[F(x), y] = 0 \quad \text{for all} \ \ x, y \in R$$

thus

$$[F(x), x] = 0 \quad \text{for all} \ \ x \in R$$

hence the commutativity of R follows from ([13], Theorem 3).
Now if $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then according to Lemma 2.1 one obtains $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.
Putting ys for y in equation (2), we get

$$[F(x), d(y)] - [F(y^*), d(x^*)] = [x, y] - [y^*, x^*] \quad \text{for all} \ \ x, y \in R . \tag{5}$$

Comparing equations (2) and (5) we arrive at

$$[F(x), d(y)] = [x, y] \quad \text{for all} \ \ x, y \in R .$$

In view of ([3], Theorem 2.11), one can conclude that R is commutative.
Further, if $[F(x), d(x^*)] = -[x, x^*]$ for all $x \in R$, then using the same techniques as used above with necessary modifications we get the required result.
(1) $\Longrightarrow$ (3) Suppose that

$$[F(x), d(x^*)] = 0 \quad \text{for all} \ \ x \in R . \tag{6}$$

Linearizing equation (6), we get $[F(x), d(y^*)] + [F(y), d(x^*)] = 0$ so that

$$[F(x), d(y)] + [F(y^*), d(x^*)] = 0 \quad \text{for all} \ \ x, y \in R . \tag{7}$$

Substituting yh for y in (7), where $h \in Z(R) \cap H(R)\backslash\{0\}$ and using (7) one obtains

$$([F(x), y] + [y^*, d(x^*)])d(h) = 0 \quad \text{for all} \ \ x, y \in R . \tag{8}$$

Since R is prime, then it follows that either $[F(x), y] + [y^*, d(x^*)] = 0$ or $d(h) = 0$. Suppose that

$$[F(x), y] + [y^*, d(x^*)] = 0 \quad \text{for all} \ \ x, y \in R . \tag{9}$$

Replacing y by ys in (9), where $s \in Z(R) \cap S(R) \backslash \{0\}$, it's obvious to verify that

$$[F(x), y] - [y^*, d(x^*)] = 0 \quad \text{for all } x, y \in R . \tag{10}$$

Adding relations (9) and (10) we obtain

$$[F(x), y] = 0 \quad \text{for all } x, y \in R$$

in such a way that

$$[F(x), x] = 0 \quad \text{for all } x \in R .$$

Therefore the commutativity of R follows from ([13], Theorem 3).
Now if $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then Lemma 2.1 yields that $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.
Writing ys instead of y in equation (7), one obtains

$$[F(x), d(y)] - [F(y^*), d(x^*)] = 0 \quad \text{for all } x, y \in R . \tag{11}$$

From (7) and (11) one obtains

$$[F(x), d(y)] = 0 \quad \text{for all } x, y \in R . \tag{12}$$

In view of ([3], Theorem 2.6), one can conclude that R is commutative. □

Remark 2.1. *To prove* (1) $\Longleftrightarrow$ (3) *the condition* $d \neq 0$ *is not necessary. Indeed, if* $F = 0$ *or* $d = 0$ *then* R *is normal and the required result follows from ([10], Lemma 2.1).*

Corollary 2.1. *Let* $(R, *)$ *be a* 2*-torsion free prime ring with involution of the second kind. If* R *admits a generalized derivation* F *associated with a nonzero derivation* d*, then the following assertions are equivalent:*
(1) $[F(x), d(y)] = \pm[x, y]$ *for all* $x, y \in R$.
(2) $[F(x), d(y)] = 0$ *for all* $x, y \in R$.
(3) R *is commutative.*

It is natural to ask what can we say about the commutativity of R if the commutator in the preceding theorem is replaced by anti-commutator. In the following theorem, we have investigated this problem for (1) $\Longleftrightarrow$ (3) and our result is of a different kind. Indeed, we have proved that the commutativity cannot be characterized by the same condition on anti-commutator.

Theorem 2.2. *Let* $(R, *)$ *be a* 2*-torsion free prime ring with involution of the second kind. There is no generalized derivation* F *associated with a derivation* d *satisfying* $F(x) \circ d(x^*) = \pm x \circ x^*$ *for all* $x \in R$.

Proof. Assume that R admits a generalized derivation F associated with a derivation d such that

$$F(x) \circ d(x^*) = x \circ x^* \quad \text{for all } x \in R . \tag{13}$$

If $d = 0$, then (13) reduces to $x \circ x^* = 0$ for all $x \in R$ which gives

$$x \circ y^* + y \circ x^* = 0 \text{ for all } x, y \in R .$$

Replacing y by h where h is a nonzero element in $Z(R) \cap H(R)$, one obtains $x + x^* = 0$. Similarly, putting s for y where s is a nonzero element in $Z(R) \cap S(R)$, we find that $x - x^* = 0$. Comparing the last two equations we conclude that $R = \{0\}$, a contradiction. Accordingly, we may suppose that $d \neq 0$. Now linearizing equation (13) one obtains $F(x) \circ d(y^*) + F(y) \circ d(x^*) = x \circ y^* + y \circ x^*$ so that

$$F(x) \circ d(y) + F(y^*) \circ d(x^*) = x \circ y + y^* \circ x^* \quad \text{for all } x, y \in R . \tag{14}$$

Replacing y by yh in (14), where $h \in Z(R) \cap H(R)\backslash\{0\}$, we arrive at

$$\big(F(x) \circ y + y^* \circ d(x^*)\big)d(h) = 0 \quad \text{for all } x, y \in R . \tag{15}$$

In light of primeness, the above expression assures that either $d(h) = 0$ or $F(x) \circ y + y^* \circ d(x^*) = 0$.
Suppose that

$$F(x) \circ y + y^* \circ d(x^*) = 0 \quad \text{for all } x, y \in R . \tag{16}$$

Putting $y = h$ where $h \in Z(R) \cap H(R)\backslash\{0\}$, we obtain $2\big(F(x) + d(x^*)\big)h = 0$ which leads to $F(x) + d(x^*) = 0$ for all $x \in R$.

On the other hand, replacing y by s in (16) with $s \in Z(R) \cap S(R)\backslash\{0\}$, we arrive at $F(x) - d(x^*) = 0$ for all $x \in R$. Therefore $F(x) = 0$ for all $x \in R$ and a fortiori $d = 0$, a contradiction.

Now suppose that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, from Lemma 2.1 it follows that $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.
Substituting ys for y in (14) for a nonzero element s in $Z(R) \cap S(R)$, one can see that

$$F(x) \circ d(y) - F(y^*) \circ d(x^*) = x \circ y - y^* \circ x^* \quad \text{for all } x, y \in R . \tag{17}$$

Adding equations (17) and (14), we conclude that

$$F(x) \circ d(y) = x \circ y \quad \text{for all } x, y \in R . \tag{18}$$

So the commutativity of R follows from ([3], Theorem 2.7). Hence equation (18) reduces to

$$F(x)d(y) = xy \quad \text{for all } x, y \in R .$$

Replacing y by yt in the last equation, we get $F(x)yd(t) = 0$ and thus

$$F(x)Rd(t) = \{0\} \quad \text{for all } x, y \in R$$

so that $d = 0$. However, in this cases $xy = 0$ for all $x, y \in R$ so that $R = \{0\}$, a contradiction.

Using similar arguments as in the first case, it is obvious to show that there is no generalized derivation F associated with a derivation d satisfying $F(x) \circ d(x^*) = -(x \circ x^*)$ for all $x \in R$. □

Corollary 2.2. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying $F(x) \circ d(y) = \pm x \circ y$ for all $x, y \in R$.*

Now if we replace the commutator by anti-commutator in (2) of Theorem 2.1 then it does not constitute a commutativity criterion. However, we can classify generalized derivations satisfying the hypothesis $F(x) \circ d(x^*) = 0$ for all $x \in R$.

Theorem 2.3. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and F a generalized derivation associated with a derivation d. Then the following assertions are equivalent:*
(1) $F(x) \circ d(x^*) = 0$ for all $x \in R$.
(2) F is a left multiplier.

Proof. We need only prove that (1) $\Longrightarrow$ (2). We are given that

$$F(x) \circ d(x^*) = 0 \quad \text{for all} \ \ x \in R\,. \tag{19}$$

Suppose that F is not a left multiplier that is $d \neq 0$.
Linearizing(19) we get

$$F(x) \circ d(y^*) + F(y) \circ d(x^*) = 0 \quad \text{for all} \ \ x, y \in R\,,$$

so that

$$F(x) \circ d(y) + F(y^*) \circ d(x^*) = 0 \quad \text{for all} \ \ x, y \in R\,. \tag{20}$$

Replacing y by yh in (20), where $h \in Z(R) \cap H(R)\backslash\{0\}$, we obtain

$$\big(F(x) \circ y + y^* \circ d(x^*)\big)d(h) = 0 \quad \text{for all} \ \ x, y \in R\,. \tag{21}$$

Invoking primeness, it follows that either $F(x) \circ y + y^* \circ d(x^*) = 0$ or $d(h) = 0$.
Suppose that

$$F(x) \circ y + y^* \circ d(x^*) = 0 \quad \text{for all} \ \ x, y \in R\,. \tag{22}$$

Taking $y = h$ where $h \in Z(R) \cap H(R)\backslash\{0\}$, one can easily see that $2\big(F(x) + d(x^*)\big)h = 0$ and thus

$$F(x) + d(x^*) = 0 \quad \text{for all} \ \ x \in R\,.$$

Replacing y by s in (22) where s is a nonzero element in $Z(R) \cap S(R)$, we obviously get

$$F(x) - d(x^*) = 0 \quad \text{for all} \ \ x \in R\,.$$

Comparing the last two relations, we conclude that $F = d = 0$, a contradiction. Therefore $d(h) = 0$ for all $h \in Z(R) \cap H(R)$ in which case Lemma 2.1 forces $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.

Putting ys for y in (20) where $s \in Z(R) \cap S(R)\backslash\{0\}$, one obtains

$$F(x) \circ d(y) - F(y^*) \circ d(x^*) = 0 \quad \text{for all} \ \ x, y \in R\,. \tag{23}$$

Adding equations (23) and (20) we conclude that

$$F(x) \circ d(y) = 0 \quad \text{for all} \quad x, y \in R\,. \tag{24}$$

Hence by ([3], Theorem 2.5) we conclude that R is commutative. Then equation (24) becomes

$$F(x)d(y) = 0 \quad \text{for all} \quad x, y \in R$$

in consequence of which $d = 0$, a contradiction. □

Corollary 2.3. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and F a generalized derivation associated with a derivation d. Then the following assertions are equivalent:*
(1) $F(x) \circ d(y) = 0$ for all $x, y \in R$.
(2) F is a left multiplier.

Theorem 2.4. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying $[F(x), d(x^*)] = \pm x \circ x^*$ for all $x \in R$.*

Proof. If R is commutative or $d = 0$, then we get $x \circ x^* = 0$ for all $x \in R$. Using the same argument as before we have $R = \{0\}$. Hence we may suppose that neither R is commutative nor $d = 0$.
Assume that

$$[F(x), d(x^*)] = x \circ x^* \quad \text{for all} \quad x \in R\,. \tag{25}$$

Linearizing (25), we get $[F(x), d(y^*)] + [F(y), d(x^*)] = x \circ y^* + y \circ x^*$ so that

$$[F(x), d(y)] + [F(y^*), d(x^*)] = x \circ y + y^* \circ x^* \quad \text{for all} \quad x, y \in R\,. \tag{26}$$

Substituting yh for y in (26), where $h \in Z(R) \cap H(R) \backslash \{0\}$, we obtain

$$\big([F(x), y] + [y^*, d(x^*)]\big)d(h) = 0 \quad \text{for all} \quad x, y \in R\,. \tag{27}$$

Since R is prime, then either $d(h) = 0$ or $[F(x), y] + [y^*, d(x^*)] = 0$.
Suppose that

$$[F(x), y] + [y^*, d(x^*)] = 0 \quad \text{for all} \quad x, y \in R\,. \tag{28}$$

this is just equation (3) in the proof of Theorem (2.1) it follows that R is commutative, a contradiction.

Now if $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then $d(s) = 0 \quad \text{for all} \quad s \in Z(R) \cap S(R)$, by Lemma 2.1.

On the other hand, replacing y by ys in (26), one obtains

$$[F(x), d(y)] - [F(y^*), d(x^*)] = x \circ y - y^* \circ x^* \quad \text{for all} \quad x, y \in R\,. \tag{29}$$

Adding equations (26) and (29) we find that

$$[F(x), d(y)] = x \circ y \quad \text{for all} \quad x, y \in R\,. \tag{30}$$

Hence by ([5], Theorem 2.2) we conclude that R is commutative, a contradiction.
The proof of the case $[F(x), d(x^*)] = -x \circ x^*$ for all $x \in R$ is similar and requires only slight modifications. ☐

Corollary 2.4. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying $[F(x), d(y)] = \pm x \circ y$ for all $x, y \in R$.*

Theorem 2.5. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F(x) \circ d(x^*) = \pm[x, x^*]$ for all $x \in R$ then R is commutative. Furthermore, F is a left multiplier.*

Proof. Assume that

$$F(x) \circ d(x^*) = [x, x^*] \quad \text{for all} \quad x \in R\,. \tag{31}$$

If $F = 0$ then the preceding relation reduces to $[x, x^*] = 0$ for all $x \in R$ hence R is commutative by ([10], Lemma 2.1). Therefore, we may assume that $F \neq 0$. Linearizing (31), one obtains

$$F(x) \circ d(y^*) + F(y) \circ d(x^*) = [x, y^*] + [y, x^*] \quad \text{for all} \quad x, y \in R \tag{32}$$

thereby

$$F(x) \circ d(y) + F(y^*) \circ d(x^*) = [x, y] + [y^*, x^*] \quad \text{for all} \quad x, y \in R\,. \tag{33}$$

Substituting yh for y in (33) where h is a nonzero element in $Z(R) \cap H(R)$ and using (33), we find that

$$\big(F(x) \circ y + y^* \circ d(x^*)\big)d(h) = 0 \quad \text{for all} \quad x, y \in R\,. \tag{34}$$

Using the primeness hypothesis, it follows that $F(x) \circ y + y^* \circ d(x^*) = 0$ or $d(h) = 0$.
Suppose that

$$F(x) \circ y + y^* \circ d(x^*) = 0 \quad \text{for all} \quad x, y \in R\,. \tag{35}$$

Putting $y = h$ in (35) where $h \in Z(R) \cap H(R)\backslash\{0\}$, one obtains

$$F(x) + d(x^*) = 0 \quad \text{for all} \quad x \in R\,.$$

Similarly, replacing y by s in (35) where $s \in Z(R) \cap S(R)\backslash\{0\}$, we arrive at

$$F(x) - d(x^*) = 0 \quad \text{for all} \quad x \in R\,.$$

Comparing the two last relations we conclude that $F = 0$, a contradiction. Thus we need only consider $d(h) = 0$ for all $h \in Z(R) \cap H(R)$. Replacing y by ys in (33) where s is a nonzero element in $Z(R) \cap S(R)$, because of $d(s) = 0$, we have

$$\big(F(x) \circ d(y) - F(y^*) \circ d(x^*)\big)s = \big([x, y] + [y^*, x^*]\big)s \quad \text{for all} \quad x, y \in R \tag{36}$$

so the primeness yields

$$F(x) \circ d(y) - F(y^*) \circ d(x^*) = [x, y] - [y^*, x^*] \quad \text{for all} \;\; x, y \in R\,. \tag{37}$$

Combining equations (33) and (37), we get

$$F(x) \circ d(y) = [x, y] \quad \text{for all} \;\; x, y \in R\,. \tag{38}$$

Hence we conclude that R is commutative by ([5], Theorem 2.5). Accordingly, equation (38) becomes $F(x)d(y) = 0$ which leads to $d = 0$, proving that F is a left multiplier. □

Corollary 2.5. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F(x) \circ d(y) = \pm[x, y]$ for all $x, y \in R$ then R is commutative. Furthermore, F is a left multiplier.*

The following example proves that the primeness hypothesis in Theorems 2.1 and 2.5 is not superfluous.

Example 1. *Let us consider $R = M_2(\mathbb{Z})$ and define $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Then F is a left multiplier and $(R, *)$ is a prime ring with involution of the first kind such that $[x, x^*] = 0 \; \forall \, x \in R$.*
Set $\mathcal{R} = R \times \mathbb{C}$, then it is obvious to verify that $(\mathcal{R}, \sigma)$ is a semi-prime ring with involution of the second kind where $\sigma(r, z) = (r^*, \bar{z})$.
Moreover, if we put

$$\mathcal{F}(r, z) = (F(r), 0)$$

then $\mathcal{F}$ is a left multiplier satisfying the conditions of Theorems 2.1, and 2.5 but $\mathcal{R}$ is not commutative.

The following example proves that the condition "* is of the second kind" is necessary in Theorems 2.1, and 2.5.

Example 2. *Let us consider $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.*
It is straightforward to check that $(R, *)$ is a prime ring with involution of the first kind such that

$$[x, x^*] = 0 \;\; \textit{for all} \;\; x \in R\,.$$

Furthermore, the mapping $F\colon R \longrightarrow R$ defined by

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is a left multiplier that satisfies conditions of Theorems 2.1 and 2.5 however R is not commutative.

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On (n, d)-Krull property in amalgamated algebra

Abstract: In this paper, we investigate the (n, d)-Krull property in the amalgamated algebra $A \bowtie^f J$. As an application, we construct examples of $(0, 0)$-Krull rings and $(1, 0)$-Krull rings having infinite dimension, and we give new classes of $(1, d)$-Krull rings which are neither $(0, d)$-Krull rings ($d \geq 0$) nor $(1, d-1)$-Krull rings ($d \geq 1$).

Keywords: Amalgamated algebra; Amalgamated duplication; Krull dimension; (n, d)-Krull ring; n-presentation; Von Neumann regular ring.

1 Introduction

Throughout this paper, all rings are commutative with identity element, and all modules are unitary. Let A and B be two rings, let J be an ideal of B and let $f: A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) / a \in A, j \in J\}$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [6, 7]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [5, 8, 9] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [6, Examples 2.5 & 2.6] and other classical constructions, such as the Nagata's idealization and the CPI extensions (in the sense of Boisen and Sheldon [1]) are strictly related to it (see [6, Example 2.7 & Remark 2.8]). See for instance [2, 3, 5, 6, 7, 8, 9, 10].

Let R be a commutative ring. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \ldots F_1 \to F_0 \to E \to 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules.

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Let A be a ring, E be an A-module, and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E (also called the idealization of E over A). Recall that $Spec(A \propto E) = \{P \propto E/P \in Spec(A)\}$.

For non-negative integers n and d, we say that a ring R is an (n, d)-Krull ring if any n-presented prime ideal P of R has height at most d; that is $ht_R(P) \leqslant d$. Note that if $dimR = d$ (where $dimR$ is the Krull dimension of R), then R is an (n, d)-Krull ring for each integer n, and if $n' \geq n$ and $d' \geq d$, then every (n, d)-Krull ring is also an (n', d')-Krull ring.

This notion generalizes, in some way, the Krull dimension. In particular, for Noetherian rings, the (n, d)-Krull property coincides with the finite Krull dimension. Apart from the Noetherian settings, the importance of this classification lies in its ability to classify some rings with infinite Krull dimension (see [11]).

In [11], the author asks whether there are examples of (n, d)-Krull rings which are neither $(n, d-1)$-Krull rings nor $(n-1, d)$-Krull rings for all non-negative integers n and d? Some limitations are immediate, for $d = 0$ or $n = 0$, the conjecture reduces to $(n, 0)$-Krull ring not $(n-1, 0)$-Krull ring or $(0, d)$-Krull ring not $(0, d-1)$-Krull ring. Again in [11], the author made use to trivial extensions to solve the case $n \leq 1$ and d arbitrary. Later, in [12], the author construct a class of $(2, d)$-Krull rings which are neither $(2, d-1)$-Krull rings (for d=1) nor $(1, d)$-Krull rings for $(d = 0, 1)$. Finally, in [13], the authors give a new examples of non-coherent domains which are $(0, d)$-Krull domains, not $(0, d-1)$-Krull domains, and where the Krull dimension equals $d + 1$.

The purpose of this work is to give new examples of $(0, 0)$-Krull rings and $(1, 0)$-Krull rings having infinite dimension, and classes of $(1, d)$-Krull rings which are neither $(0, d)$-Krull rings $(d \geq 0)$ nor $(1, d-1)$-Krull rings $(d \geq 1)$.

2 Construction of (n, d)-Krull rings having infinite dimensions

In this section, we study the amalgamated algebra $A \bowtie^f J$ to be $(0, 0)$-Krull ring. The goal is to provide examples of $(0, 0)$-Krull rings having infinite dimension.

First, it is worthwhile noting that the function $f^n : A^n \to B^n$ defined by $f^n((\alpha_i)_{i=1}^{i=n}) = (f(\alpha_i))_{i=1}^{i=n}$ is a ring homomorphism, $(A \bowtie^f J)^n \cong A^n \bowtie^{f^n} J^n$ and $f^n(\alpha a) = f(\alpha)f^n(a)$ for all $\alpha \in A$ and $a \in A^n$

Next, before we announce the main result of this section (Theorem 2.1), we make the following useful remark.

Remark 2.1. *Let (A, M) be a local ring, $f : A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $J \subseteq Rad(B)$. Then:*
1. *$A \bowtie^f J$ is a local ring and $M \bowtie^f J$ is its maximal ideal.*
2. *$f(A \backslash M) + J \subseteq U(B)$.*

Proof. **(1)** Indeed, by [7, Proposition 2.6 (5)], $Max(A \bowtie^f J) = \{m \bowtie^f J / m \in Max(A)\} \cup \{\overline{Q}\}$ with $Q \in Max(B) \setminus V(J)$ and $\overline{Q} := \{(a, f(a) + j) / a \in A, j \in J, f(a) + j \in Q\}$. Since $J \subseteq Rad(B)$ then, $J \subseteq Q$ for all $Q \in Max(B)$. So, $Max(A \bowtie^f J) = \{m \bowtie^f J / m \in Max(A)\} = M \bowtie^f J$ since (A, M) is a local ring. Therefore $(A \bowtie^f J, M \bowtie^f J)$ is a local ring.
(2) Let $x \in A \backslash M, j \in J$. Then $f(x) + j = f(x)(1 + jf(x^{-1})) \in U(B)$ since $f(x) \in U(B)$ and $1 + jf(x^{-1}) \in U(B)$, as desired. □

Next we announce the first main result of this paper.

Theorem 2.1. *Let $f : A \to B$ be a ring homomorphism, and let J be a proper ideal of B.*
1. *Assume that $J \subseteq Nil(B)$. If A is a $(0, 0)$-Krull ring, then so is $A \bowtie^f J$.*
2. *Assume that either $f(Spec(A)) \subseteq J$ or (A, M) is a local ring such that $f(M)J = 0$.*
   (a) *Assume that $J \subseteq Nil(B)$ and J is not finitely generated ideal of $f(A) + J$. Then $A \bowtie^f J$ is a $(0, 0)$-Krull ring.*
   (b) *Assume that J is not finitely generated ideal of $f(A) + J$ and $Spec(B) \subseteq (f(A) + J)$. If $f(A) + J$ is a $(0, 0)$-Krull ring, then so is $A \bowtie^f J$.*

Before proving Theorem 2.1, we establish the following Lemma.

Lemma 2.1. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B, and let U be an $(f(A) + J)$-module such that $U \subseteq (f(A) + J)^n$. Then*
1. *$\overline{U} := \{(a, f^n(a) + k) / a \in A^n, k \in J^n, f^n(a) + k \in U\} \subseteq A^n \bowtie^{f^n} J^n$ is an $(A \bowtie^f J)$-module.*
2. *If $\overline{U}$ is a finitely generated $(A \bowtie^f J)$-module, then U is a finitely generated $(f(A) + J)$-module.*

Proof. 1. It is obvious.
2. Assume that $\overline{U} = \sum_{i=1}^{i=p} A \bowtie^f J(a_i, f^n(a_i) + k_i)$, where $a_i \in A^n$, $k_i \in J^n$ and $f^n(a_i) + k_i \in U$ for each $i = 1, \dots, p$. Then, $\sum_{i=1}^{i=p}(f(A) + J)(f^n(a_i) + k_i) \subseteq U$. Conversely, let $u = f^n(a) + k \in U$ for some $a \in A^n$ and $k \in J^n$. Then, $(a, f^n(a) + k) = \sum_{i=1}^{i=p}(\alpha_i, f(\alpha_i) + j_i)(a_i, f^n(a_i) + k_i) \in \overline{U}$, where $(\alpha_i, f(\alpha_i) + j_i) \in A \bowtie^f J$ for all i. So, $u = \sum_{i=1}^{i=p}(f(\alpha_i) + j_i)(f^n(a_i) + k_i) \in \sum_{i=1}^{i=p}(f(A) + J)(f^n(a_i) + k_i)$. Thus, $U = \sum_{i=1}^{i=p}(f(A) + J)(f^n(a_i) + k_i)$, as desired. □

Proof of Theorem 2.1: (1) Assume that $J \subseteq Nil(B)$. By [7, Proposition 2.6 (5)], $Spec(A \bowtie^f J) = \{P \bowtie^f J / P \in Spec(A)\} \cup \{\overline{Q}\}$ with $Q \in Spec(B) \backslash V(J)$, where $V(J) := \{P \in Spec(B) / P \supseteq J\}$ and $\overline{Q} := \{(a, f(a) + j) / a \in A, j \in J, f(a) + j \in Q\}$. Since $J \subseteq Nil(B)$, then $J \subseteq Q$ for all $Q \in Spec(B)$. So, $Spec(A \bowtie^f J) = \{P \bowtie^f J / P \in Spec(A)\}$.

Assume that A is a $(0, 0)$-Krull ring, and let $K := P \bowtie^f J$ be a finitely generated prime ideal of $A \bowtie^f J$. Clearly, P is a finitely generated prime ideal of A. So, $ht_A(P) = 0$ since A is a $(0, 0)$-Krull ring, and so $ht_{A \bowtie^f J}(P \bowtie^f J) = 0$. Therefore $A \bowtie^f J$ is a $(0, 0)$-Krull ring.
(2) Assume that either $f(Spec(A)) \subseteq J$ or (A, M) is a local ring such that $f(M)J = 0$.

(a) Assume that $J \subseteq Nil(B)$ and J is not finitely generated ideal of $f(A) + J$. We claim that each prime ideal of $A \bowtie^f J$ is not finitely generated. Otherwise, there exists a finitely generated prime ideal of $A \bowtie^f J$, $K := P \bowtie^f J := \sum_{i=1}^{i=n}(A \bowtie^f J)(p_i, f(p_i) + k_i)$, where P is a prime ideal of A, $p_i \in P$, and $k_i \in J$ for all $1 \leq i \leq n$.

Assume that $f(Spec(A)) \subseteq J$, and let $j \in J$. Then $(0, j) = \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i)+j_i)(p_i, f(p_i)+k_i)$ for some $\alpha_i \in A$ and $j_i \in J$. So, $j = \sum_{i=1}^{i=n}(f(\alpha_i)+j_i)(f(p_i)+k_i) \in \sum_{i=1}^{i=n}(f(A)+J)(f(p_i)+k_i)$. Thus $J = \sum_{i=1}^{i=n}(f(A) + J)(f(p_i) + k_i)$ since $f(p_i) \in J$ for all $1 \leq i \leq n$, a contradiction.

Assume that (A, M) is a local ring such that $f(M)J = 0$, and let $k \in J$. Then, $(0, k) = \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i) + j_i)(p_i, f(p_i) + k_i)$ for some $\alpha_i \in A$ and $j_i \in J$. So $\sum_{i=1}^{i=n} \alpha_i p_i = 0$, and $k = \sum_{i=1}^{i=n}(f(\alpha_i) + j_i)(f(p_i) + k_i) = \sum_{i=1}^{i=n}(f(\alpha_i) + j_i)k_i \in \sum_{i=1}^{i=n}(f(A) + J)k_i$ (since $f(P)J = 0$). Thus $J = \sum_{i=1}^{i=n}(f(A) + J)k_i$, a contradiction. Hence, each prime ideal of $A \bowtie^f J$ is not finitely generated and so $A \bowtie^f J$ is a $(0, 0)$-Krull ring.

(b) Assume that J is not finitely generated ideal of $f(A) + J$, $Spec(B) \subseteq (f(A) + J)$, and $f(A) + J$ is a $(0, 0)$-Krull ring. Let K be a proper finitely generated prime ideal of $A \bowtie^f J$. Then $K = \overline{Q}$, where $Q \in Spec(B)\backslash V(J)$. So, Q is a finitely generated prime ideal of $f(A) + J$ by Lemma 2.1 (2). Hence, $ht_A(Q) = 0$ since $f(A) + J$ is a $(0, 0)$-Krull ring. It is easy to show that $ht_{A\bowtie^f J}(\overline{Q}) = 0$. Otherwise, there is K' a prime ideal of $A \bowtie^f J$ such that $K' \subsetneq K$.

If $K' = P \bowtie^f J$, where $P \in Spec(A)$, then $J \subseteq Q$, a contradiction. So $K' = \overline{Q'}$, where $Q' \in Spec(B)\backslash V(J)$. Let $u = f(a) + j \in Q'$, where $a \in A$, $j \in J$. Then $(a, f(a) + j) \in K' \subseteq K$. So, $u = f(a) + j \in Q$, and so $Q' \subseteq Q$. Moreover, there is $(a, f(a) + j) \in K \setminus K'$. So $f(a) + j \in Q \setminus Q'$. Thus, $Q' \subsetneq Q$, the desired contradiction, and this complete the proof of Theorem 2.1.

The following Corollaries are immediate consequences of Theorem 2.1.

Corollary 2.1. *Let A be a ring and let I be a proper ideal of A.*
1. *Assume that $I \subseteq Nil(A)$. If A is a $(0, 0)$-Krull ring, then so is $A \bowtie I$.*
2. *Assume that (A, M) is a local ring such that $IM = 0$.*
   (a) *Assume that $I \subseteq Nil(A)$ and I is not finitely generated ideal of A. Then $A \bowtie I$ is a $(0, 0)$-Krull ring.*
   (b) *Assume that I is not finitely generated ideal of A. If A is a $(0, 0)$-Krull ring, then so is $A \bowtie I$.*

Corollary 2.2. *Let A be a ring such that $Spec(A) = I$ be not finitely generated ideal of A. Then $A \bowtie I$ is a $(0, 0)$-Krull ring.*

Now, we are able to give classes of $(0, 0)$-Krull rings having infinite dimension.

Example 1. *Let (R, m) be a local ring such that $dimR = \infty$. Let $A := R \propto (\frac{R}{m})^\infty$ and $M = m \propto (\frac{R}{m})^\infty$ its maximal ideal, $B := A \propto E$, where E be an $\frac{A}{M}$-vector space, and let $J := Nil(A) \propto E$ be a proper ideal of B. Consider the ring homomorphism $f: A \to B$ ($f(a) = (a, 0)$). Then*
1. *$dim(A \bowtie^f J) = Max\{dimA, dim(f(A) + J)\} = dimA = \infty$ by Proposition [7, Proposition 4.1].*
2. *$A \bowtie^f J$ is a $(0, 0)$-Krull by Theorem 2.1 (1) since A is by Theorem [11, Theorem 2.3].*

Example 2. *Let $A = K[[X_1, \ldots, X_n, \ldots]]$ be the formal series ring of infinite indeterminates $(X_i)_{i=1,\ldots,\infty}$ over a field K, M its maximal ideal. Let $B := A \propto (\frac{A}{M})^\infty$ be trivial*

extension ring of A by $(\frac{A}{M})^{\infty}$, $J := M \propto (\frac{A}{M})^{\infty}$, and consider the ring homomorphism $f: A \to B$ ($f(x) = (a, 0)$). Then:

1. *$dim(A \bowtie^f J) = dim(A) = \infty$ by Proposition [7, Proposition 4.1].*
2. *$A \bowtie^f J$ is a $(0, 0)$-Krull ring by Theorem 2.1 (2) (b) since for all $P \in Spec(A)$, $f(P) = P \propto 0 \subseteq J$ and $f(A) + J = A \propto (\frac{A}{M})^{\infty} = B$ is a $(0, 0)$-Krull ring by Theorem [11, Theorem 2.3].*

Example 3. *Let (R, m) be a local ring such that $dimR = \infty$ and $A := R \propto (\frac{R}{m})^{\infty}$. Then*

1. *$dim(A \bowtie Nil(A)) = dimA = \infty$.*
2. *$A \bowtie Nil(A)$ is $(0, 0)$-Krull ring by Corollary 2.1 (1) since A is by Theorem [11, Theorem 2.3].*

Now we establish the second main result of this section.

Theorem 2.2. *Let $f: A \to B$ be a ring homomorphism, and let J be a proper ideal of B.*

1. *Assume that $J \subseteq Nil(B)$. If A is a Von-Neumann regular ring, then $A \bowtie^f J$ is an $(0, 0)$-Krull ring.*
2. *Assume that J is not finitely generated ideal of $(f(A) + J)$ and $Spec(B) \subseteq (f(A) + J)$. If $f(A) + J$ is a Von-Neumann regular ring, then $A \bowtie^f J$ is an $(0, 0)$-Krull ring.*

Proof. **(1)** Assume that $J \subseteq Nil(B)$. Then $Spec(A \bowtie^f J) = \{P \bowtie^f J / P \in Spec(A)\}$. Let $P \bowtie^f J$ be a finitely generated ideal of $(A \bowtie^f J)$, where P is a proper prime ideal of A. Clearly, P is a finitely generated ideal of A. Consider the exact sequence of A-modules

$$0 \to P \to A \to \frac{A}{P} \to 0$$

The cyclic A-module $\frac{A}{P}$ is a projective since it is a finitely presented A-module and A is a $(1, 0)$-ring. Thus, the above exact sequence splits. So, $P = Ae$ is generated by an idempotent element $e \in P$ ($\neq 0$). Our aim is to show that $ht_{A \bowtie^f J}(P \bowtie^f J) = 0$. Otherwise, there is P' a prime ideal of A such that $P' \bowtie^f J \subsetneq P \bowtie^f J$. So, $P' \subsetneq P$. But, $e(e-1) = 0 \in P'$. So $e \in P'$ or $e - 1 \in P'$ since P' is prime ideal of A. Assume that $e - 1 \in P' \subseteq P$. Then $1 = (1 - e) + e \in P$, absurd since P is a proper ideal of A. Hence $e \in P'$ and so $P' = P$, the desired contradiction.

(2) Assume that J is not finitely generated ideal of $(f(A)+J)$ and $Spec(B) \subseteq (f(A)+J)$. Let $\overline{Q}$ be a finitely generated ideal of $(A \bowtie^f J)$, where Q is a proper prime ideal of $f(A)+J$. Then Q is a finitely generated ideal of $f(A) + J$ by Lemma 2.1. Consider the exact sequence of $(f(A) + J)$-modules

$$0 \to Q \to f(A) + J \to \frac{f(A) + J}{Q} \to 0$$

Since the cyclic $(f(A) + J)$-module, $\frac{f(A)+J}{Q}$ is a finitely presented and $f(A) + J$ is a $(1, 0)$-ring, then $\frac{f(A)+J}{Q}$ is a projective. Thus, the above exact sequence splits. So, $Q = (f(A) + J)f$ is generated by an idempotent element $f \in Q$ ($\neq 0$). Therefore, $ht_{A \bowtie^f J}(\overline{Q}) = 0$. Otherwise, there is Q' a prime ideal of $f(A) + J$ such that $\overline{Q'} \subsetneq \overline{Q}$. So, $Q' \subsetneq Q$. But, $f(f-1) = 0 \in Q'$ implies that $f \in Q'$, and so $Q' = Q$, a contradiction, and this completes the proof of Theorem 2.2. □

We make the following corollary.

Corollary 2.3. *Let A be a ring and let I be a proper ideal of A. Assume that either $I \subseteq Nil(A)$ or I is not finitely generated ideal of A. If A is a Von-Neumann regular ring, then $A \bowtie I$ is an $(0,0)$-Krull ring.*

Example 4. *Let A be a non-Noetherian Von Neumann regular ring and let I be a non finitely generated ideal of A (see for example [4, Example 2.7]). Then $A \bowtie^f I$ is an $(0,0)$-Krull ring by corollary 2.3.*

Example 5. *Let A be a Von-Neumann regular ring, I be a proper ideal of A, and let $n \geq 2$ such that $I^n \neq 0$. Let $B := \frac{A}{I^n}$, and consider the ring homomorphism $f\colon A \to B$ ($f(a) = \overline{a}$). Then $A \bowtie^f \overline{I}$ is an $(0,0)$-Krull ring by Theorem 2.2 (1).*

3 Construction of (n,d)-Krull rings which are neither $(n,d-1)$-Krull rings nor $(n-1,d)$-Krull rings

In this section, we investigate the amalgamated algebra $A \bowtie^f J$ to be $(1,0)$-Krull ring. The aim is to give new classes of $(1,d)$-Krull rings which are neither $(1,d-1)$-Krull rings nor $(0,d)$-Krull rings.

Theorem 3.1. *Let (A,M) be a local ring, $f\colon A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $f(M)J = 0$ and $J \subseteq Nil(B)$. Assume that either A is a $(1,0)$-Krull ring or M is not finitely generated ideal of A or J is not finitely generated ideal of $f(A)+J$. Then $A \bowtie^f J$ is a $(1,0)$-Krull ring.*

Proof. Assume that A is a $(1,0)$-Krull ring. Let $K := P \bowtie^f J := \sum_{i=1}^{i=n}(A \bowtie^f J)(p_i, f(p_i)+k_i)$ be a finitely presented ideal of $A \bowtie^f J$, where P is a prime ideal of A, $p_i \in P$, and $k_i \in J$ for all $1 \leq i \leq n$. Then $P = \sum_{i=1}^{i=n} p_i$ and $J = \sum_{i=1}^{i=n}(f(A)+J)k_i$. We may assume that $\{(p_i)_{i=1}^{i=n}\}$ is a minimal generating set of P, and $\{(k_i)_{i=1}^{i=n}\}$ is a minimal generating set of J. Consider the exact sequence of A-modules:

$$0 \to Kerv \to A^n \to P \to 0 \tag{1}$$

where $v((\alpha_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n} \alpha_i p_i$. On the other hand consider the exact sequence of $(f(A)+J)$-modules:

$$0 \to Keru \to (f(A)+J)^n \to J \to 0 \tag{2}$$

where $u((f(\alpha_i)+j_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n}(f(\alpha_i)+j_i)k_i$. So,

$$\begin{aligned} Keru &= \{(f(\alpha_i)+j_i)_{i=1,\dots,n} \in (f(A)+J)^n \diagup \sum_{i=1}^{i=n}(f(\alpha_i)+j_i)k_i = 0\} \\ &\cong f^n(M^n) + L_n\,. \end{aligned}$$

(by Remark 2.1 since $\{(k_i)_{i=1}^{i=n}\}$ is a minimal generating set of J), where $L_n = \{(j_i)_{i=1}^{i=n} \in J^n / \sum_{i=1}^{i=n} j_i k_i = 0\}$. Consider the exact sequence of $(A \bowtie^f J)$-modules:

$$0 \to Kerw \to (A \bowtie^f J)^n \to P \bowtie^f J \to 0 \tag{3}$$

where $w((\alpha_i, f(\alpha_i) + j_i)_{i=1,\ldots,n}) = \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i) + j_i)(p_i, f(p_i) + k_i) = \sum_{i=1}^{i=n}(\alpha_i p_i, f(\alpha_i p_i) + (f(\alpha_i) + j_i)k_i)) = (\sum_{i=1}^{i=n} \alpha_i p_i, \sum_{i=1}^{i=n} f(\alpha_i p_i) + \sum_{i=1}^{i=n} f(\alpha_i)k_i + \sum_{i=1}^{i=n} k_i j_i)$. It follows that $Kerw \cong Kerv \bowtie^{f^n} L_n$ (since $Kerv \subseteq M^n$). By a sequence (3), $Kerw$ is a finitely generated $(A \bowtie^f J)$-module. Thus, $Kerv$ is a finitely generated A-module. Therefore P is a finitely presented ideal of A by a sequence (1). So, $ht_A(P) = 0$ since A is a $(1, 0)$-Krull ring, and so $ht_{A \bowtie^f J}(P \bowtie^f J) = 0$. Thus $A \bowtie^f J$ is a $(1, 0)$-Krull ring.

Assume that J is not finitely generated ideal of $f(A) + J$. Then $A \bowtie^f J$ is a $(0, 0)$-Krull ring by Theorem 2.1 (2) (a), and so, $A \bowtie^f J$ is a $(1, 0)$-Krull ring.

Assume that M is a not finitely generated ideal of A. We claim that each prime ideal of $A \bowtie^f J$ is not finitely presented. Indeed, assume that there exists a finitely presented ideal $K := P \bowtie^f J$ of $A \bowtie^f J$, where P is a prime ideal of A. Then P and J are finitely generated ideals of A and $f(A) + J$ respectively. Let $\{(p_i)_{i=1}^{i=n}\}$ is a minimal generating set of P, and $\{(k_i)_{i=1}^{i=m}\}$ is a minimal generating set of J. Consider the exact sequence of A-modules:

$$0 \to Kerv \to A^n \to P \to 0 \tag{4}$$

where $v((\alpha_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n} \alpha_i p_i$. On the other hand consider the exact sequence of $(f(A) + J)$-modules:

$$0 \to Keru \to (f(A) + J)^m \to J \to 0 \tag{5}$$

where $u((f(\alpha_i) + j_i)_{i=1,\ldots,m}) = \sum_{i=1}^{i=m}(f(\alpha_i) + j_i)k_i$. So,

$$\begin{aligned} Keru &= \{(f(\alpha_i) + j_i)_{i=1,\ldots,m} \in (f(A) + J)^m / \sum_{i=1}^{i=m}(f(\alpha_i) + j_i)k_i = 0\} \\ &\cong f^m(M^m) + L_m \ . \end{aligned}$$

(by Remark 2.1 since $\{(k_i)_{i=1}^{i=m}\}$ is a minimal generating set of J), where $L_m = \{(j_i)_{i=1}^{i=m} \in J^m / \sum_{i=1}^{i=m} j_i k_i = 0\}$. Consider the exact sequence of $(A \bowtie^f J)$-modules:

$$0 \to Kerw \to (A \bowtie^f J)^{n+m} \to P \bowtie^f J \to 0 \tag{6}$$

where

$$\begin{aligned} & w((\alpha_i, f(\alpha_i) + j_i)_{i=1,\ldots,n}, (\beta_i, f(\beta_i) + g_i)_{i=1,\ldots,m}) \\ = \ & \textstyle\sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i) + j_i)(p_i, f(p_i)) + \sum_{i=1}^{i=m}(\beta_i, f(\beta_i) + g_i)(0, k_i) \\ = \ & (\textstyle\sum_{i=1}^{i=n} \alpha_i p_i, \sum_{i=1}^{i=n} f(\alpha_i p_i) + \sum_{i=1}^{i=m}(f(\beta_i) + g_i)k_i \ . \end{aligned}$$

It follows that $Kerw \cong Kerv \bowtie^{f^n} J^n \bigoplus M^m \bowtie^{f^m} L_m$. Therefore $Kerw$ is not finitely generated $(A \bowtie^f J)$-module since M is not finitely generated ideal of A), a contradiction. Hence, each prime ideal of $A \bowtie^f J$ is not finitely presented and so $A \bowtie^f J$ is a $(1, 0)$-Krull ring, and this complete the proof of Theorem 3.1. □

As a consequence of Theorem 3.1, we establish the following Corollary.

Corollary 3.1. *Let (A, M) be a local ring, and let I be a proper ideal of A such that $IM = 0$ and $I \subseteq Nil(A)$. Assume that either A is a $(1,0)$-Krull ring or M or I is not finitely generated ideal of A. Then $A \bowtie^f I$ is a $(1,0)$-Krull ring.*

Theorem 3.1 enriches the literature with new examples of $(1, d)$-Krull rings which are neither $(1, d-1)$-Krull rings nor $(0, d)$-Krull rings.

Example 6. *Let $A = K[[X_1, \ldots, X_n, \ldots]]$ be the formal series ring of infinite indeterminates $(X_i)_{i=1,\ldots,\infty}$ over a field K and M its maximal ideal. Let E be an $\frac{\frac{A}{M^2}}{\frac{M}{M^2}}$-vector space, $B := \frac{A}{M^2} \propto \frac{A}{M^2}.e$, where $e \in E(\neq 0)$, $J = \frac{A}{M^2}\overline{m} \propto \frac{A}{M^2}.e$, where $m \in M$, and consider the ring homomorphism $f\colon A \to B$ ($f(a) = (\overline{a}, 0)$). Let A_2 be a Noetherian ring such that $dimA_2 = d$, and set $A_1 := A \bowtie^f J$ and $C := A_1 \times A_2$ the direct product of A_1 and A_2. Then:*

1. *$dim(A \bowtie^f J) = Max\{dimA, dim\frac{A}{M^2}\} = dimA = \infty$ by Proposition [7, Proposition 4.1].*
2. *$A \bowtie^f J$ is a $(1,0)$-Krull ring by Theorem 3.1 since M is not finitely generated ideal of A.*
3. *$A \bowtie^f J$ is not $(0, d)$-Krull ring for each positive integer d. Indeed, assume that $A \bowtie^f J$ is $(0, d)$-Krull ring for a positive integer d. Let P be a prime ideal of A generated by $(X_i)_{i=1,\ldots,d+1}$. Then $Q = P \bowtie^f J$ is a prime ideal of $A \bowtie^f J$. So, $ht_{A\bowtie^f J}(Q) = ht_A(P) = d + 1$. But Q is a finitely generated prime ideal of $A \bowtie^f J$ (generated by $(X_i, (\overline{X_i}, 0))_{i=1,\ldots,d+1}, (0, (\overline{m}, e))$), a contradiction since $A \bowtie^f J$ is $(0, d)$-Krull ring. Hence $A \bowtie^f J$ is not $(0, d)$-Krull ring for each positive integer d.*
4. *C is a $(1, d)$-Krull ring since A_i is a $(1, d)$-Krull ring for each $i = 1, 2$ by (2) and Theorem [11, Theorem 2.1 & Theorem 2.6].*
5. *C is not $(0, d)$-Krull ring since A_1 is not $(0, d)$-Krull ring by (3) and Theorem [11, Theorem 2.6].*
6. *C is not a $(1, d-1)$-Krull ring. Assume that C is a $(1, d-1)$-Krull ring. Hence A_2 is a $(1, d-1)$-Krull ring and so $dimA_2 \leq d - 1$ by Theorem [11, Theorem 2.1], a contradiction since $dimA_2 = d$. Therefore C is not a $(1, d-1)$-Krull ring.*

Example 7. *Let (R, m) be a local and $n \geq 2$ such that m is not finitely generated ideal of R and $m^n \neq 0$. Let $A := \frac{R}{m^n} \propto (\frac{\frac{R}{m^n}}{\frac{m}{m^n}})^\infty$ and $M = \frac{m}{m^n} \propto (\frac{\frac{R}{m^n}}{\frac{m}{m^n}})^\infty$ its maximal ideal. Let $I := \frac{m^{n-1}}{m^n} \propto (\frac{\frac{R}{m^n}}{\frac{m}{m^n}})^\infty$. Then $A \bowtie I$ is a $(1,0)$-Krull ring by Corollary 3.1 since M is not finitely generated ideal of A.*

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Noor Mohammad Khan and Ahsan Mahboob

Pure ideals in ordered Γ-semigroups

Abstract: We introduce the notions of pure ideals, left(right) weakly pure ideals and purely prime ideals in an ordered Γ-semigroup. We, then, define a right weakly regular ordered Γ-semigroup and characterize right weakly regular ordered Γ-semigroups in terms of its ideals, bi-ideals and interior ideals.

Keywords: Ordered Γ-semigroup; pure ideal; right weakly regular ordered Γ-semigroup.

1 Introduction and Preliminaries

In 1986, M.K. Sen and N.K. Saha introduced the notion of a Γ-semigroup. Later on in 1993, the notion of an ordered Γ-semigroup was introduced by M.K. Sen and A. Seth. Many classical notions of ordered Γ-semigroups have been studied by [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and [13]. In 2009, Bashir and Shabir [1] introduced the notions of pure ideal, weakly pure ideal and purely prime ideal in ternary semigroup. Later on in 2014, J. Sanborisoot and T. Changphas [12] defined the concept of pure ideals, weakly pure ideals and purely prime ideals in an ordered ternary semigroups. Motivated by these, we introduce the concepts of pure ideals, weakly pure ideals and purely prime ideals in ordered Γ-semigroups.

Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping from $S\times\Gamma\times S$ to S which maps $(a, \alpha, b) \to a\alpha b$ satisfying $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.

Let S and Γ be non-empty sets. Then the triplet $(S, \Gamma, \leq)$ is called an ordered Γ-semigroup if S is a Γ-semigroup and $(S, \leq)$ is a partially ordered set such that $a \leq b \Rightarrow a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

For any subsets A and B of an ordered Γ-semigroup S, denote

$$A\Gamma B = \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$$
$$\text{and } (A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}\ .$$

Let S be an ordered Γ-semigroup. A non-empty subset T of S is said to be Γ-subsemigroup of S if for all $x, y \in T$ and $\gamma \in \Gamma$, $x\gamma y \in T$. A non-empty subset A of S called idempotent, if $A = (A\Gamma A]$. A non-empty subset A of S is called left(right) ideal of S if $S\Gamma A \subseteq A(A\Gamma S \subseteq A)$ and $a \in A$, $S \ni b \leq a \Rightarrow b \in A$. A non-empty subset J

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of S is called an ideal of S if J is both a left ideal and a right ideal of S. A Γ-sub-semigroup B of an ordered Γ-semigroup S is called a bi-ideal of S if $B\Gamma S\Gamma B \subseteq B$ and $a \in B,\ S \ni b \leq a \Rightarrow b \in B$. It is easy to verify that the intersection of any family of ideals of an ordered Γ-semigroup S is either empty or an ideal of S and union of any family of ideals of an ordered Γ-semigroup S is an ideal of S. An ordered Γ-semigroup S is called regular (left regular, right regular) if for each $x \in S$ there exist $y \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq x\alpha y\beta x (x \leq y\alpha x\beta x,\ x \leq x\alpha x\beta y)$.

Let S be an ordered Γ-semigroup and let A be any non-empty subset of S. Then by $L(A),\ R(A),\ J(A)$, and $B(A)$, we denote the left ideal of S generated by A, the right ideal of S generated by A, the ideal of S generated by A and the bi-ideal of S generated by A respectively. It is easy to show that $L(A) = (A \cup S\Gamma A],\ R(A) = (A \cup A\Gamma S],\ J(A) = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$ and $B(A) = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$.

If $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$, $L(a)$ instead of $L(\{a\})$, the left ideal of S, $R(a)$ instead of $R(\{a\})$ the right ideal of S, $J(a)$ instead of $J(\{a\})$ the ideal of S and $B(a)$ instead of $B(\{a\})$ the bi-ideal of S generated by a respectively in the sequel.

Lemma 1.1. *Let S be an ordered Γ-semigroup and A, B be any non-empty subsets of S. Then followings hold*
(1) $A \subseteq (A]$;
(2) If $A \subseteq B$, *then* $(A] \subseteq (B]$;
(3) $((A]] = (A]$;
(4) $(A]\Gamma(B] \subseteq (A\Gamma B]$;
(5) If L is left ideal and R a right ideal of S, then the set $(L\Gamma R]$ is an ideal of S;
(6) If $A,\ B$ are ideals of S, then $(A\Gamma B],\ (B\Gamma A],\ A \cup B,\ A \cap B$ are ideals of S;
(7) $(S\Gamma a]((a\Gamma S], (S\Gamma a\Gamma S])$ is a left ideal (resp. right right, ideal) of S for every $a \in S$;
(8) $((A]\Gamma(B]] = (A\Gamma B]$.

2 Pure ideals

Definition 2.1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. An ideal A of S is called a left(right) pure ideal of S if for each $x \in A$, there exist $y \in A$ and $\gamma \in \Gamma$ such that $x \leq y\gamma x$ ($x \leq x\gamma y$).*

Theorem 2.1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and let A be an ideal of S. Then A is a right pure ideal if and only if $B \cap A = (B\Gamma A]$ for each right ideal B of S.*

Proof. Assume that A is a right pure ideal of the ordered Γ-semigroup S. Take any right ideal B of S. Then $B\Gamma A \subseteq B\Gamma S \subseteq B$. Therefore, by Lemma 1.1, $(B\Gamma A] \subseteq (B] = B$. Since $B\Gamma A \subseteq S\Gamma A \subseteq A$ implies $(B\Gamma A] \subseteq (A] = A$. Therefore $(B\Gamma A] \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$. By assumption, there exist $y \in A$ and $\gamma \in \Gamma$ such that $x \leq x\gamma y$. Since $x\gamma y \in B\Gamma A$. Therefore $x \in (B\Gamma A]$. Thus $B \cap A \subseteq (B\Gamma A]$.

Conversely suppose that $B \cap A = (B\Gamma A]$ for each right ideal B of S. Let $x \in A$. Since $(x \cup x\Gamma S]$ is right ideal of S and $S\Gamma A \subseteq A$. By assumption

$$\begin{aligned}(x \cup x\Gamma S] \cap A &= ((x \cup x\Gamma S]\Gamma A] = ((x \cup x\Gamma S]\Gamma(A]] = ((x \cup x\Gamma S)\Gamma A]\\ &= (x\Gamma A \cup x\Gamma S\Gamma A] \subseteq (x\Gamma A]\end{aligned}$$

Since $x \in (x \cup x\Gamma S] \cap A$, $x \in (x\Gamma A]$. Hence A is a right pure ideal of the ordered Γ-semigroup S. □

Theorem 2.2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with 0. Then*
(i) $\{0\}$ is right pure ideal of S;
(ii) Union of any family of right pure ideals of S is a right pure ideal of S;
(iii) Finite intersection of right pure ideals of S is a right pure ideal of S.

Proof. (i). obvious.
(ii). Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be any family of right pure ideals of S. As union of ideals is an ideal, $\bigcup_{\lambda\in\Lambda} A_\lambda$ is an ideal of S. Let $x \in \bigcup_{\lambda\in\Lambda} A_\lambda$. Then there exist some $\lambda \in \Lambda$ such that $x \in A_\lambda$. Since A_λ is right pure ideal, there exist $y \in A_\lambda$ and $\gamma \in \Gamma$ such that $x \leq x\gamma y$. Now $y \in \bigcup_{\lambda\in\Lambda} A_\lambda$ and $\gamma \in \Gamma$ such that $x \leq x\gamma y$. Hence $\bigcup_{\lambda\in\Lambda} A_\lambda$ is a right pure ideal of the ordered Γ-semigroup S.
(iii). Let $A_1, A_2 A_3, \ldots, A_n$ be right pure ideals of S. Then $\cap_{i=1}^{n} A_i$ is an ideal of S. Let $x \in \cap_{i=1}^{n} A_i$, for k=1,2,3,.....,n there exist $y_k \in A_k$ and $\gamma_k \in \Gamma$ such that $x \leq x\gamma_k y_k$ i.e. $x \leq x\gamma_1 y_1, x \leq x\gamma_2 y_2 \ldots\ldots\ldots x \leq x\gamma_n y_n$. Therefore $x \leq x\gamma_1 y_1 \leq x\gamma_2 y_2 \gamma_1 y_1 \leq \ldots\ldots\ldots \leq x\gamma_n y_n \gamma_{n-1} y_{n-1} \ldots\ldots \gamma_1 y_1 \leq x\gamma_n(y_n \gamma_{n-1} y_{n-1} \ldots\ldots \gamma_1 y_1)$. Since $y_n \gamma_{n-1} y_{n-1} \ldots\ldots \gamma_1 y_1 \in \cap_{i=1}^{n} A_i$. Therefore $\cap_{i=1}^{n} A_i$ is a right pure ideal of S. □

Theorem 2.3. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with 0 and let A be an ideal of S. Then A contains the largest right pure ideal of S denoted by $\mathcal{S}(A)$. It is called the pure part of A.*

Proof. Since 0 is a right pure ideal of the ordered Γ-semigroup S contained in A, it follows that union of all right pure ideals of S contained in A exists and is the largest right pure ideal of S contained in A. □

Proposition 2.1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with 0. Let A, B and A_λ, $(\lambda \in \Lambda)$ be ideals of the ordered Γ-semigroup S. Then*
(i) $\mathcal{S}(A \cap B) = \mathcal{S}(A) \cap \mathcal{S}(B)$;
(ii) $\bigcup_{\lambda\in\Lambda} \mathcal{S}(A_\lambda) \subseteq \mathcal{S}(\bigcup_{\lambda\in\Lambda} A_\lambda)$.

Proof. (i). Since $\mathcal{S}(A) \subseteq A$ and $\mathcal{S}(B) \subseteq B$, $\mathcal{S}(A) \cap \mathcal{S}(B) \subseteq A \cap B$. But $\mathcal{S}(A) \cap \mathcal{S}(B)$ is right pure ideal, so $\mathcal{S}(A) \cap \mathcal{S}(B) \subseteq \mathcal{S}(A \cap B)$. As $\mathcal{S}(A \cap B) \subseteq A \cap B \subseteq A$ and $\mathcal{S}(A \cap B)$ is right pure ideal, $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A)$. Similarly $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(B)$. Thus $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$. Hence $\mathcal{S}(A \cap B) = \mathcal{S}(A) \cap \mathcal{S}(B)$.

(ii). Since $\mathcal{S}(A_\lambda) \subseteq A_\lambda$, so $\underset{\lambda\in\Lambda}{\cup} \mathcal{S}(A_\lambda) \subseteq \underset{\lambda\in\Lambda}{\cup} A_\lambda$. As $\mathcal{S}(A_\lambda)$ is right pure, $\underset{\lambda\in\Lambda}{\cup} \mathcal{S}(A_\lambda)$ is also right pure ideal. Thus $\underset{\lambda\in\Lambda}{\cup} \mathcal{S}(A_\lambda) \subseteq \mathcal{S}(\underset{\lambda\in\Lambda}{\cup} A_\lambda)$. □

Definition 2.2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then a right pure ideal A is called purely maximal if for any proper right pure ideal B of S, $A \subseteq B \Rightarrow A = B$.*

Definition 2.3. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then a right pure ideal I is called purely prime if $I_1 \cap I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ for any right pure ideals I_1 and I_2 of S.*

Proposition 2.2. *Any purely maximal ideal of an ordered Γ-semigroup $(S, \Gamma, \leq)$ is purely prime.*

Proof. Suppose A is purely maximal ideal of S and A_1, A_2 are right pure ideals of S such that $A_1 \cap A_2 \subseteq A$, Suppose $A_1 \nsubseteq A$. Then $A_1 \cup A$ is a right pure ideal of S such that $A \subsetneq A_1 \cup A$. Since A is purely maximal. So $A_1 \cup A = S$. Thus $A_2 = A_2 \cap S = A_2 \cap (A_1 \cup A) = (A_2 \cap A_1) \cup (A_2 \cap A) \subseteq A \cup A = A$. Hence A is purely prime. □

Proposition 2.3. *The pure part of any maximal ideal of an ordered Γ-semigroup $(S, \Gamma, \leq)$ with 0 is purely prime.*

Proof. Let A be a maximal ideal of an ordered Γ-semigroup $(S, \Gamma, \leq)$ and $\mathcal{S}(A)$ be its pure part. Suppose $A_1 \cap A_2 \subseteq \mathcal{S}(A)$, where A_1, A_2 are right pure ideals of S. If $A_1 \subseteq A$, then $A_1 \subseteq \mathcal{S}(A)$. If $A_1 \nsubseteq \mathcal{S}(A)$, then $A_1 \nsubseteq A$. As A is maximal, $A_1 \cup A = S$. Now

$$\begin{aligned} A_2 &= A_2 \cap S = A_2 \cap (A_1 \cup A) \\ &= (A_2 \cap A_1) \cup (A_2 \cap A) \\ &\subseteq \mathcal{S}(A) \cup A \subseteq A \cup A = A\,. \end{aligned}$$

As A_2 is right pure and $\mathcal{S}(A)$ is the largest right pure ideal contained in A, $A_2 \subseteq \mathcal{S}(A)$. Hence $\mathcal{S}(A)$ is purely prime. □

Proposition 2.4. *Let I be a right pure ideal of an ordered Γ-semigroup $(S, \Gamma, \leq)$ and $a \in S$ such that $a \notin I$. Then there exist a purely prime ideal K of S such that $I \subseteq K$ and $a \notin K$.*

Proof. Let $X = \{J \mid J$ is right pure ideal of S, $I \subseteq J$ and $a \notin J\}$. As $I \subseteq I$ and $a \notin I$, $I \in X$. X is partially ordered by inclusion. Let $\{J_\lambda\}_{\lambda\in\Lambda}$ be any totally ordered subset of X. As union of right pure ideals is right pure, $\underset{\lambda\in\Lambda}{\cup} J_\lambda$ is right pure ideal. Since $I \subseteq \underset{\lambda\in\Lambda}{\cup} J_\lambda$ and $a \notin \underset{\lambda\in\Lambda}{\cup} J_\lambda$, so $\underset{\lambda\in\Lambda}{\cup} J_\lambda \in X$. Thus, by Zorn's Lemma, X has a maximal element, say, K such that K is right pure, $I \subseteq K$ and $a \notin K$. We claim that K is purely prime. Suppose I_1 and I_2 are right pure ideals of an ordered Γ-semigroup S such that $I_1 \nsubseteq K$ and $I_2 \nsubseteq K$. Since I_1, I_2 and K are right pure, $I_1 \cup K$ and $I_2 \cup K$ are right pure ideals of S such that $K \subsetneq I_1 \cup K$ and $K \subsetneq I_1 \cup K$. Thus $a \in I_1 \cup K, I_2 \cup K$. As $a \notin K$, $a \in I_1 \cap I_2$. Hence $I_1 \cap I_2 \nsubseteq K$. This shows that K is purely prime ideal of S. □

Proposition 2.5. *Any ideal I of an ordered Γ-semigroup $(S, \Gamma, \leq)$ is the intersection of all the purely prime ideals of S containing I.*

Proof. By Proposition 2.4, there exists a purely prime ideal containing I. Let $\{J_\lambda\}_{\lambda \in \Lambda}$ be the family of all purely prime ideals of the ordered Γ-semigroup S which contain I. Since $I \subseteq J_\lambda$ for each $\lambda \in \Lambda$, $I \subseteq \bigcap_{\lambda \in \Lambda} J_\lambda$. To show that $\bigcap_{\lambda \in \Lambda} J_\lambda \subseteq I$. Suppose on contrary that $\bigcap_{\lambda \in \Lambda} J_\lambda \nsubseteq I$. Let $a \in \bigcap_{\lambda \in \Lambda} J_\lambda$ such that $a \notin I$. Then, by Proposition 2.4 there exists a purely prime ideal J such that $I \subseteq J$ and $a \notin J$. It follows that $a \notin \bigcap_{\lambda \in \Lambda} J_\lambda$ which is a contradiction. Therefore $\bigcap_{\lambda \in \Lambda} J_\lambda \subseteq I$. Hence $\bigcap_{\lambda \in \Lambda} J_\lambda = I$. □

3 Weakly pure ideals in ordered Γ-semigroups

Definition 3.1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. An ideal A of S is called left(right) weakly pure if $A \cap B = (A\Gamma B]$ $(A \cap B = (B\Gamma A])$ for each ideal B of S.*

It is easy to see that in an ordered Γ-semigroup, every left (right) pure ideal is left (right) weakly pure.

Proposition 3.1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with 0. If A and B are ideals of S, then*

$$BA^{-1} = \{s \in S \mid x\gamma s \in B \text{ for all } x \in A, \gamma \in \Gamma\}$$

and

$$A_{-1}B = \{s \in S \mid s\gamma x \in B \text{ for all } x \in A, \gamma \in \Gamma\}$$

are ideals of S.

Proof. We have to show that BA^{-1} is an ideal of S. Clearly $0 \in BA^{-1}$. Let $u \in S$, $\gamma \in \Gamma$ and $s \in BA^{-1}$. To show that $u\gamma s \in BA^{-1}$, take any $x \in A$. Since $x\alpha u \in A$ for all $\alpha \in \Gamma$, we have $x\alpha(u\gamma s) = (x\alpha u)\gamma s \in B$. Thus $u\gamma s \in BA^{-1}$. Similarly $s\gamma u \in BA^{-1}$.
Let $x \in BA^{-1}$ and $y \in S$ be such that $y \leq x$. Now for any $z \in A$, since $z\gamma y \leq z\gamma x$ for $\gamma \in \Gamma$ and $z\gamma x \in B$, we have $z\gamma y \in B$. Hence $y \in BA^{-1}$. Therefore BA^{-1} is an ideal of S.

Similarly we may show that $A_{-1}B$ is an ideal of S. □

Theorem 3.1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and let A be an ideal of S. Then A is a left(a right) weakly pure ideal if and only if $BA^{-1} \cap A = A \cap B(A_{-1}B \cap A = A \cap B)$ for all ideals B of S.*

Proof. Suppose that A is left weakly pure ideal and let B be an ideal of S. By Proposition 3.1, BA^{-1} is an ideal of S. Thus $A \cap BA^{-1} = (A\Gamma(BA^{-1})]$. Since $A\Gamma(BA^{-1}) \subseteq A\Gamma S \subseteq A$, we have $(A\Gamma(BA^{-1})] \subseteq (A] = A$. Let $t \in (A\Gamma(BA^{-1})]$. Then $t \leq x\gamma y$ for some $x \in A$, $\gamma \in \Gamma$ and $y \in BA^{-1}$. By definition of BA^{-1}, $x\gamma y \in B$. Thus $t \in B$. This implies that $A \cap BA^{-1} =$

$(A\Gamma(BA^{-1})] \subseteq A \cap B$. For the reverse inclusion, take any $a \in A \cap B$. Since $x\gamma a \in B$ for any $x \in A,\ \gamma \in \Gamma$, we have $a \in BA^{-1}$. Thus $a \in BA^{-1} \cap A$ and so $A \cap B \subseteq BA^{-1} \cap A$.

Conversely suppose that $BA^{-1} \cap A = A \cap B$ for each ideal B of S. To show that A is left weakly pure, we show that for any ideal C of S, $A \cap C = (A\Gamma C]$. By assumption, $A \cap C = CA^{-1} \cap A$. Since $A\Gamma C \subseteq A\Gamma S \subseteq A$, $(A\Gamma C] \subseteq (A] = A$. Let $t \in (A\Gamma C]$, then $t \leq x\gamma y$ for some $x \in A,\ \gamma \in \Gamma$ and $y \in C$. Now for any $a \in A,\ \alpha \in \Gamma$, $a\alpha(x\gamma y) \in C$ and so $x\gamma y \in CA^{-1}$. Therefore $t \in CA^{-1}$. Thus $(A\Gamma C] \subseteq CA^{-1}$. Now $(A\Gamma C] \subseteq CA^{-1} \cap A = A \cap C$. For the reverse inclusion take any $c \in C,\ a \in A$ and $\gamma \in \Gamma$. Now $a\gamma c \in A\Gamma C \subseteq (A\Gamma C]$. This implies that $c \in (A\Gamma C]A^{-1}$. Therefore $C \subseteq (A\Gamma C]A^{-1}$. Now $A \cap C \subseteq (A\Gamma C]A^{-1} \cap A = A \cap (A\Gamma C] \subseteq (A\Gamma C]$. Therefore $(A\Gamma C] = A \cap C$. □

Definition 3.2. *An ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be right weakly regular if for any $x \in S,\ x \in ((x\Gamma S)\Gamma(x\Gamma S)]$ or equivalently $A \subseteq (A\Gamma S\Gamma A\Gamma S]$ for each $A \subseteq S$.*

It is easy to check that every regular ordered Γ-semigroup is right weakly regular.

Definition 3.3. *An ordered Γ-semigroup S is said to be commutative if $A\Gamma B = B\Gamma A$ for each $A, B \subseteq S$.*

Proposition 3.2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. If S is commutative right weakly regular then it is regular.*

Proof. Let S be a commutative right weakly regular ordered Γ-semigroup. Let A be any subset of S. Therefore $A \subseteq (A\Gamma S\Gamma A\Gamma S] = (A\Gamma S\Gamma S\Gamma A] \subseteq (A\Gamma S\Gamma A]$. Hence S is regular ordered Γ-semigroup. □

Definition 3.4. *A Γ-subsemigroup I of an ordered Γ-semigroup S is said to be an interior ideal of S if $S\Gamma I\Gamma S \subseteq I$ and $a \in I, S \ni b \leq a \Rightarrow b \in I$. It is easy to show that $I(A) = (A \cup A\Gamma A \cup S\Gamma A\Gamma S]$ and $I(a) = (a \cup a\Gamma a \cup S\Gamma a\Gamma S]$ are the interior ideals generated by the subset A and an element a of S respectively.*

Proposition 3.3. *Let $(S, \Gamma, \leq)$ be a right weakly regular ordered Γ-semigroup. Then every interior ideal of S is an ideal of S.*

Proof. Let I be an interior ideal of a right weakly regular ordered Γ-semigroup S. Therefore $I\Gamma S \subseteq (I\Gamma S\Gamma I\Gamma S]\Gamma S = (I\Gamma S\Gamma I\Gamma S]\Gamma(S] \subseteq (I\Gamma S\Gamma I\Gamma S\Gamma S] \subseteq (I\Gamma I\Gamma S] \subseteq (S\Gamma I\Gamma S] \subseteq (I] \subseteq I$. Also $S\Gamma I \subseteq S\Gamma(I\Gamma S\Gamma I\Gamma S] = (S]\Gamma(I\Gamma S\Gamma I\Gamma S] \subseteq (S\Gamma I\Gamma S\Gamma I\Gamma S] \subseteq (S\Gamma I\Gamma S] \subseteq (I] \subseteq I$. Hence I is an ideal of S. □

Corollary 3.1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. If S is right weakly regular, then interior ideals and ideals coincide.*

Theorem 3.2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then the followings are equivalent:*
(i) S is right weakly regular;
(ii) Each right ideal of S is idempotent;
(iii) $A \cap B = (B\Gamma A]$ for each right ideal B and ideal A of S.

Proof. $(i) \Rightarrow (ii)$. Assume that S is right weakly regular. Let A be a right ideal of a Γ-semigroup S. Since $A\Gamma A \subseteq A\Gamma S \subseteq S \subseteq A$, we have $(A\Gamma A] \subseteq A$. Let $x \in A$. By assumption, $x \in ((x\Gamma S)\Gamma(x\Gamma S)] \subseteq (A\Gamma A]$. Then $A \subseteq (A\Gamma A]$. Therefore $(A\Gamma A] = A$
$(ii) \Rightarrow (i)$. Suppose that every right ideal of S is idempotent. Let $x \in S$. Then $(x \cup x\Gamma S]$ is the right ideal of S, so idempotent i.e;

$$\begin{aligned}(x \cup x\Gamma S] &= ((x \cup x\Gamma S]\Gamma(x \cup x\Gamma S]] \\ &= ((x \cup x\Gamma S)\Gamma(x \cup x\Gamma S)] \\ &= (x\Gamma x \cup x\Gamma x\Gamma S \cup x\Gamma S\Gamma x \cup x\Gamma S\Gamma x\Gamma S] \\ &\subseteq ((x\Gamma S)\Gamma(x\Gamma S)]\ .\end{aligned}$$

As $x \in (x \cup x\Gamma S]$, $x \in ((x\Gamma S)\Gamma(x\Gamma S)]$. Hence S is right weakly regular.
$(i) \Rightarrow (iii)$. Suppose S is right weakly regular ordered Γ-semigroup and B be a right ideal and A an ideal of S. Then $B\Gamma A \subseteq B\Gamma S \subseteq B$. This implies that $(B\Gamma A] \subseteq B$. Similarly $(B\Gamma A] \subseteq A$. Then $(B\Gamma A] \subseteq A \cap B$. Let $x \in B \cap A$. We have $((x\Gamma S)\Gamma(x\Gamma S)] \subseteq (B\Gamma A]$. By assumption, we get $x \in ((x\Gamma S)\Gamma(x\Gamma S)]$. Hence $x \in (B\Gamma A]$. Thus $B \cap A \subseteq (B\Gamma A]$. Hence $B \cap A = (B\Gamma A]$.
$(iii) \Rightarrow (i)$. Assume that $B \cap A = (B\Gamma A]$ for all right ideals B and all ideals A of S. To prove that S is right weakly regular, take any $x \in S$. Then $B = (x \cup x\Gamma S]$ and $A = (x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S]$ are right ideal and ideal of S respectively. So, by hypothesis

$$\begin{aligned}x \in A \cap B &= (B\Gamma A] \\ &= ((x \cup x\Gamma S]\Gamma(x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S]] \\ &= ((x \cup x\Gamma S)\Gamma(x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S)] \\ &\subseteq (x\Gamma x \cup x\Gamma x\Gamma S \cup x\Gamma S\Gamma x \cup x\Gamma S\Gamma x\Gamma S]\ .\end{aligned}$$

Therefore $x \le x\alpha x$ or $x \le x\alpha x\beta a$ or $x \le x\alpha a\beta x$ or $x \le x\alpha a\beta x\gamma b$ for some $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. If $x \le x\alpha x$, then $x \le x\alpha x\alpha x\alpha x$. If $x \le x\alpha x\beta a$, then $x \le x\alpha x\alpha x\beta(a\beta a)$. If $x \le x\alpha a\beta x$, then $x \le x\alpha a\beta x\alpha a\beta x = x\alpha a\beta x\alpha(a\beta x)$. In all cases $x \in ((x\Gamma S)\Gamma(x\Gamma S)]$. Hence S is right weakly regular ordered Γ-semigroup. □

Theorem 3.3. *Let $(S, \Gamma, \le)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if $B \cap I \subseteq (B\Gamma I]$ for each bi-ideal B and each interior ideal I of S.*

Proof. Let S be a right weakly regular ordered Γ-semigroup. Let B and I are bi-ideal and interior ideal of S. Take $a \in B \cap I$. Since S is right weakly regular, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \le a\alpha x\beta a\gamma y \in B\Gamma S\Gamma I\Gamma S \subseteq B\Gamma I$. Therefore $a \in (B\Gamma I]$. Hence $B \cap I \subseteq (B\Gamma I]$.

Conversely take any $x \in S$. Let $B = (x \cup x\Gamma x \cup x\Gamma S\Gamma x]$, the bi-ideal generated by x and $I = (x \cup x\Gamma x \cup S\Gamma x\Gamma S]$, the interior ideal generated by x. By hypothesis

$$\begin{aligned}x \in B \cap I &\subseteq (B\Gamma I] \\ &= ((x \cup x\Gamma x \cup x\Gamma S\Gamma x]\Gamma(x \cup x\Gamma x \cup S\Gamma x\Gamma S]]\end{aligned}$$

$$
\begin{aligned}
&= ((x \cup x\Gamma x \cup x\Gamma S\Gamma x)\Gamma(x \cup x\Gamma x \cup S\Gamma x\Gamma S)] \\
&\subseteq (x\Gamma x \cup x\Gamma x\Gamma x \cup x\Gamma S\Gamma x \cup x\Gamma S\Gamma x\Gamma S]\ .
\end{aligned}
$$

Therefore $x \leq x\alpha x$ or $x \leq x\alpha x\beta x$ or $x \leq x\alpha a\beta x$ or $x \leq x\alpha a\beta x\gamma b$ for some $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. If $x \leq x\alpha x$, then $x \leq x\alpha x\alpha x\alpha x$. If $x \leq x\alpha x\beta x$, then $x \leq x\alpha x\beta x\alpha x\beta x = x\alpha x\beta x\alpha(x\beta x)$. If $x \leq x\alpha a\beta x$, then $x \leq x\alpha a\beta x\alpha a\beta x = x\alpha a\beta x\alpha(a\beta x)$. In all cases $x \in ((x\Gamma S)\Gamma(x\Gamma S)]$. Hence S is right weakly regular ordered Γ-semigroup. ☐

Corollary 3.2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if $B \cap A \subseteq (B\Gamma A]$ for each bi-ideal B and each ideal A of S.*

Proof. As every ideal is an interior ideal, therefore, by Theorem 3.3, $B \cap A \subseteq B\Gamma A$ for every bi-ideal B and every ideal A of S.

Conversely take any $x \in S$. Let $B = (x \cup x\Gamma x \cup x\Gamma S\Gamma x]$, the bi-ideal generated by x and $A = (x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S]$, the ideal generated by x. By hypothesis

$$
\begin{aligned}
x \in B \cap A &\subseteq (B\Gamma A] \\
&= ((x \cup x\Gamma x \cup x\Gamma S\Gamma x]\Gamma(x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S]] \\
&= ((x \cup x\Gamma x \cup x\Gamma S\Gamma x)\Gamma(x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S)] \\
&\subseteq (x\Gamma x \cup x\Gamma S\Gamma x \cup x\Gamma x\Gamma S \cup x\Gamma S\Gamma x\Gamma S]\ .
\end{aligned}
$$

Therefore $x \leq x\alpha x$ or $x \leq x\alpha a\beta x$ or $x \leq x\alpha x\beta a$ or $x \leq x\alpha a\beta x\gamma b$ for some $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. If $x \leq x\alpha x$, then $x \leq x\alpha x\alpha x\alpha x$. If $x \leq x\alpha a\beta x$, then $x \leq x\alpha a\beta x\alpha a\beta x = x\alpha a\beta x\alpha(a\beta x)$. If $x \leq x\alpha x\beta a$, then $x \leq x\alpha(x\alpha x\beta a)\beta a = x\alpha x\alpha x\beta(a\beta a)$. In all cases $x \in ((x\Gamma S)\Gamma(x\Gamma S)]$. Hence S is right weakly regular ordered Γ-semigroup. ☐

Theorem 3.4. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if $B \cap I \cap R \subseteq (B\Gamma I\Gamma R]$ for each bi-ideal B, each interior ideal I and each right ideal R of S.*

Proof. Let S be a right weakly regular, ordered Γ-semigroup, B a bi-ideal of S, I an interior ideal of S and R a right ideal of S. Let $a \in B \cap I \cap R$. Since S is right weakly regular there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that

$$
\begin{aligned}
a &\leq a\alpha x\beta a\gamma y \\
&\leq a\alpha x\beta(a\alpha x\beta a\gamma y)\beta a\gamma y \\
&= (a\alpha x\beta a)\alpha(x\beta a\gamma y)\beta(a\gamma y) \\
&\in (B\Gamma S\Gamma B)\Gamma(S\Gamma I\Gamma S)\Gamma(R\Gamma S) \subseteq B\Gamma I\Gamma R
\end{aligned}
$$

Therefore $a \in (B\Gamma I\Gamma R]$. Hence $B \cap I \cap R \subseteq (B\Gamma I\Gamma R]$.

Conversely take any $x \in S$. Let $B = (x \cup x\Gamma x \cup x\Gamma S\Gamma x]$, the bi-ideal generated by x, $I = (x \cup x\Gamma x \cup S\Gamma x\Gamma S]$, the interior ideal generated by x and $R = (x \cup x\Gamma S]$, the right

ideal of S generated by x. By hypothesis,

$$\begin{aligned} x &\in B \cap I \cap R \\ &\subseteq (B\Gamma I\Gamma R] \\ &= ((x \cup x\Gamma x \cup x\Gamma S\Gamma x]\Gamma(x \cup x\Gamma x \cup S\Gamma x\Gamma S]\Gamma(x \cup x\Gamma S]] \\ &\subseteq ((x \cup x\Gamma x \cup x\Gamma S\Gamma x]\Gamma((x \cup x\Gamma x \cup S\Gamma x\Gamma S)\Gamma(x \cup x\Gamma S)]] \\ &= ((x \cup x\Gamma x \cup x\Gamma S\Gamma x)\Gamma(x \cup x\Gamma x \cup S\Gamma x\Gamma S)\Gamma(x \cup x\Gamma S)] \\ &\subseteq (x\Gamma x\Gamma x \cup x\Gamma x\Gamma x\Gamma x \cup x\Gamma x\Gamma x\Gamma x\Gamma x \cup x\Gamma S\Gamma x \cup x\Gamma S\Gamma x\Gamma S]\ . \end{aligned}$$

Therefore $x \leq x\alpha x\beta x$ or $x \leq x\alpha x\beta x\gamma x$ or $x \leq x\alpha x\beta x\gamma x\delta x$ or $x \leq x\alpha a\beta x$ or $x \leq x\alpha a\beta x\gamma b$ for some $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. If $x \leq x\alpha x\beta x$, then $x \leq x\alpha x\beta x\alpha x\beta x = x\alpha x\beta x\alpha(x\beta x)$. If $x \leq x\alpha x\beta x\gamma x\delta x$, then $x \leq x\alpha x\beta x\gamma(x\delta x)$. If $x \leq x\alpha a\beta x$, then $x \leq x\alpha a\beta x\alpha a\beta x = x\alpha a\beta x\alpha(a\beta x)$. In all cases $x \in ((x\Gamma S)\Gamma(x\Gamma S)]$. Hence S is right weakly regular ordered Γ-semigroup. □

Corollary 3.3. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if $B \cap I \cap R \subseteq B\Gamma I\Gamma R$ for each bi-ideal B, each ideal I and each right ideal R of S.*

Proof. As every ideal is an interior ideal. Therefore, by Theorem 3.4, $B \cap I \cap R \subseteq B\Gamma I\Gamma R$.

Conversely take any $x \in S$. Let $B = (x \cup x\Gamma x \cup x\Gamma S\Gamma x]$ the bi-ideal generated by x, $I = (x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S]$ the ideal generated by x and $R = (x \cup x\Gamma S]$ the right ideal of S generated by x. By hypothesis

$$\begin{aligned} x &\in B \cap I \cap R \\ &\subseteq (B\Gamma I\Gamma R] \\ &= ((x \cup x\Gamma x \cup x\Gamma S\Gamma x]\Gamma(x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S]\Gamma(x \cup x\Gamma S]] \\ &\subseteq (((x \cup x\Gamma x \cup x\Gamma S\Gamma x)\Gamma(x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S)]\Gamma(x \cup x\Gamma S]] \\ &= ((x \cup x\Gamma x \cup x\Gamma S\Gamma x)\Gamma(x \cup x\Gamma S \cup S\Gamma x \cup S\Gamma x\Gamma S)\Gamma(x \cup x\Gamma S)] \\ &\subseteq (x\Gamma x\Gamma x \cup x\Gamma x\Gamma x\Gamma x \cup x\Gamma S\Gamma x \cup x\Gamma S\Gamma x\Gamma S]\ . \end{aligned}$$

Therefore $x \leq x\alpha x\beta x$ or $x \leq x\alpha x\beta x\gamma x$ or $x \leq x\alpha a\beta x$ or $x \leq x\alpha a\beta x\gamma b$ for some $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. If $x \leq x\alpha x\beta x$, then $x \leq x\alpha x\beta x\alpha x\beta x \leq x\alpha x\beta x\alpha(x\beta x)$. If $x \leq x\alpha a\beta x$, then $x \leq x\alpha a\beta x\alpha a\beta x = x\alpha a\beta x\alpha(a\beta x)$. In all cases $x \in ((x\Gamma S)\Gamma(x\Gamma S)]$. Hence S is right weakly regular ordered Γ-semigroup. □

Theorem 3.5. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then S is right weakly regular if and only if every ideal I of S is right pure.*

Proof. The proof follows from Theorem 2.1 and Theorem 3.2. □

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Hidetoshi Marubayashi and Akira Ueda

Projective ideals of differential polynomial rings over HNP rings

Abstract: Let R be a hereditary Noetherian prime ring with derivation δ. We describe all projective ideals of a differential polynomial ring $R[x;\delta]$ and give some examples of hereditary Noetherian prime rings R with derivation δ to show various phenomena about projective ideals in $R[x;\delta]$.

Keywords: Projective ideal; Hereditary Noetherian prime ring; Differential polynomial ring.

1 Preliminary

Let R be an HNP ring with its quotient ring Q, δ be a derivation of R and $S = R[x;\delta]$ be a differential polynomial ring in an indeterminate x.

The aim of this paper is to describe all projective ideals of S, that is, left and right projective ideals of S, which is used to show S is a G-HNP ring defined in [2].

In Section 2, first we describe δ-maximal projective ideals of R (Lemma 2.1) which is used to describe all projective ideals of S and show that any projective ideal of S is a product of an invertible ideal and an eventually idempotent projective ideal. In particular, an eventually idempotent projective ideal is of the form: $\mathfrak{a}[x;\delta]$, where $\mathfrak{a}$ is a δ-stable eventually idempotent ideal of R.
In Section 3, we provide examples of HNP rings with derivations δ in order to explain the various phenomena in Section 2.

Let R be a prime Goldie ring with its quotient ring Q. We use the following notation; for X and Y be subsets of Q, $(X : Y)_l = \{q \in Q \mid qY \subseteq X\}$ and $(X : Y)_r = \{q \in Q \mid Yq \subseteq X\}$. Let I be a fractional right R-ideal. We define $I_v = (R : (R : I)_l)_r$, a fractional right R-ideal containing I. I is called a *right v-ideal* if $I_v = I$. Similarly for any fractional left R-ideal J, define ${}_vJ = (R : (R : J)_r)_l$ which is called a *left v-ideal* if ${}_vJ = J$. A fractional R-ideal A is called a *v-ideal* if ${}_vA = A = A_v$. An integral v-ideal is referred to as a *v-ideal* of R. By a *maximal v-ideal* of R we mean a v-ideal which is maximal amongst all v-ideals of R.

We refer the reader to [10] for elementary properties for order theory.

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2 Structure of projective ideals of S

Throughout the paper, R is a hereditary Noetherian prime ring (an HNP ring for short) with derivation δ and Q is its quotient ring of R unless otherwise stated. δ is extended to a derivation of Q by $\delta(c^{-1}) = -c^{-1}\delta(c)c^{-1}$. Let $S = R[x;\delta]$ be a differential polynomial ring in an indeterminate x and $T = Q[x;\delta]$. δ is extended to a derivation of S (T) by

$$\delta(f(x)) = \delta(a_n)x^n + \cdots + \delta(a_0), \text{ where } f(x) = a_n x^n + \cdots + a_0 \in S\ (T)\,,$$

equivalently, $\delta(f(x)) = xf(x) - f(x)x$, an inner derivation induced by x.
The following elementary properties of δ-stable ideals are frequently used in the paper:

(1) Any idempotent ideal $\mathfrak{a}$ of R is δ-stable, that is, $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$.
(2) For any ideal $\mathfrak{a}$ of R, $\mathfrak{a}[x;\delta]$ is an ideal of S if and only $\mathfrak{a}$ is δ-stable.
(3) Any ideal A of S is δ-stable since δ is inner induced by x and $\mathfrak{a} = A \cap R$ is also δ-stable.
(4) For any ideal A of S, $C(A) = \{a_n \in R \mid \exists f(x) = a_n x^n + \cdots + a_0 \in A\} \cup (0)$ is δ-stable.

Definition 2.1. *A δ-stable ideal $\mathfrak{p}$ of R is called a δ-prime ideal if $\mathfrak{ab} \subseteq \mathfrak{p}$, where $\mathfrak{a}, \mathfrak{b}$ are δ-stable ideals, implies either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, which is equivalent to if $\mathfrak{ab} \subseteq \mathfrak{p}$, where $\mathfrak{a}, \mathfrak{b}$ are ideals and one of them is δ-stable, implies either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.*

Lemma 2.1. *Let $\mathfrak{p}$ be a δ-stable ideal of R. Then the following three conditions are equivalent:*
(1) $\mathfrak{p}$ is a δ-maximal ideal (ideal maximal amongst the δ-stable ideals).
(2) $\mathfrak{p}$ is a δ-prime ideal.
(3) Either $\mathfrak{p}$ is an idempotent maximal ideal or $\mathfrak{p} = \mathfrak{m}^e$ for some invertible maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^e$ is δ-stable for some $e \geq 1$ and $\mathfrak{m}^l$ is not δ-stable for all l $(1 \leq l < e)$ in case $e \geq 2$.

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (3) Suppose $\mathfrak{p}$ is a δ-prime ideal and let $\mathfrak{m}_i$ be the maximal ideals containing $\mathfrak{p}$ $(i = 1, \ldots, n)$. Then $(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n)^l \subseteq \mathfrak{p}$ for some $l \geq 1$.
(i) If $\mathfrak{m}_i$ are all idempotents, then $\mathfrak{p} = \mathfrak{m}_i$ for some i and so $\mathfrak{p}$ is an idempotent maximal ideal.
(ii) If there is some $\mathfrak{m}_i$ which is invertible, then we may assume that $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are invertible $(r \geq 1)$ and $\mathfrak{m}_{r+1}, \ldots, \mathfrak{m}_n$ are idempotent. Put $\mathfrak{a} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r = \mathfrak{m}_1 \ldots \mathfrak{m}_r$ and $\mathfrak{b} = \mathfrak{m}_{r+1} \cap \cdots \cap \mathfrak{m}_n$. By [5, Proposition 2.8], $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b} = \mathfrak{ba}$. Thus $\mathfrak{a}^l\mathfrak{b}^l = (\mathfrak{a} \cap \mathfrak{b})^l = (\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n)^l \subseteq \mathfrak{p}$ and $\mathfrak{b}^l$ is δ-stable. Hence either $\mathfrak{a}^l \subseteq \mathfrak{p}$ or $\mathfrak{b}^l \subseteq \mathfrak{p}$.
In case $\mathfrak{b}^l \subseteq \mathfrak{p}$, we have $\mathfrak{b} \subseteq \mathfrak{p} \subseteq \mathfrak{m}_{r+1}$ and so $\mathfrak{p} = \mathfrak{m}_{r+1}$.
In case $\mathfrak{a}^l \subseteq \mathfrak{p}$. If $\mathfrak{b} \neq R$, then $\mathfrak{a}^l \subseteq \mathfrak{p} \subseteq \mathfrak{m}_{r+1}$, a contradiction. If $\mathfrak{b} = R$, that is, all $\mathfrak{m}_i$ are

invertible and so $r = n$. Then, by [5, Theorem 4.2 and Proposition 2.1], $\mathfrak{p} = \mathfrak{m}_1^{e_1} \dots \mathfrak{m}_n^{e_n}$ for $e_i \geq 1$. It is enough to prove that $n = 1$. We suppose, on the contrary, that $n > 1$. For any $a_i \in \mathfrak{m}_i^{e_i}$ $(1 \leq i \leq n)$, we have

$$\mathfrak{m}_n^{e_n} \supseteq \mathfrak{p} \ni \delta(a_1 \dots a_n) = \delta(a_1 \dots a_{n-1})a_n + a_1 \dots a_{n-1}\delta(a_n)\,.$$

Thus $a_1 \dots a_{n-1}\delta(a_n) \in \mathfrak{m}^{e_n}$ and $\mathfrak{m}_1^{e_1} \dots \mathfrak{m}_{n-1}^{e_{n-1}}\delta(a_n) \subseteq \mathfrak{m}_n^{e_n}$ follows. Since $\mathfrak{m}_1^{e_1} \dots \mathfrak{m}_{n-1}^{e_{n-1}} + \mathfrak{m}_n^{e_n} = R$, we have $\delta(a_n) \in \mathfrak{m}_n^{e_n}$ and so $\delta(\mathfrak{m}_n^{e_n}) \subseteq \mathfrak{m}_n^{e_n}$, that is, $\mathfrak{m}_n^{e_n}$ is δ-stable. Hence either $\mathfrak{m}_1^{e_1} \dots \mathfrak{m}_{n-1}^{e_{n-1}} \subseteq \mathfrak{p}$ or $\mathfrak{m}_n^{e_n} \subseteq \mathfrak{p}$, which are both impossible. Hence $n = 1$, that is, $\mathfrak{p} = \mathfrak{m}^e$ for some invertible maximal ideal $\mathfrak{m}$ and some $e \geq 1$. It is easy to see that $\mathfrak{m}^l$ is not δ-stable for all l $(1 \leq l < e)$ if $e > 1$.
(3) ⇒ (1). It is enough to prove that $\mathfrak{p}$ is a δ-maximal ideal in case $\mathfrak{p} = \mathfrak{m}^e$ $(e > 1)$ and $\mathfrak{m}^l$ is not δ-stable for all $l(1 \leq l < e)$. Let $\mathfrak{a}$ be an ideal such that $R \supset \mathfrak{a} \supset \mathfrak{p}$. Since $\mathfrak{m}$ is only maximal ideal containing $\mathfrak{a}$, we may assume that $\mathfrak{m}^l \supseteq \mathfrak{a}$ and $\mathfrak{m}^{l+1} \not\supseteq \mathfrak{a}$ for some l with $e > l \geq 1$. So $R \supseteq \mathfrak{m}^{-l}\mathfrak{a}$ and if $R \supset \mathfrak{m}^{-l}\mathfrak{a}$, then $\mathfrak{m} \supseteq \mathfrak{m}^{-l}\mathfrak{a}$. Thus $\mathfrak{a} \subseteq \mathfrak{m}^{l+1}$, a contradiction. Hence $R = \mathfrak{m}^{-l}\mathfrak{a}$ and $\mathfrak{a} = \mathfrak{m}^l$ is not δ-stable. Therefore $\mathfrak{p} = \mathfrak{m}^e$ is a δ-maximal ideal. □

We use $V_r(S)$ to denote the set of all ideals of S which are right v-ideals and $V_{(m,r)}$ to denote the set of all maximal members in $V_r(S)$.

Note that $gl.dim(S) = 2$ ([10, (7.5.3)]). So an ideal A of S is in $V_r(S)$ if and only if A is right projective by [3, Proposition 5.2]. We also define $V_l(S)$ and $V_{(m,l)}(S)$ analogously. We use the notation: $Spec_0(S) = \{P$: a prime ideal of S with $P \cap R = (0)\}$. Then we have the following lemma.

Lemma 2.2. *The mapping*

$$Spec_0(S) \longrightarrow Spec(T)$$

$$P \longrightarrow PT$$

is a bijection whose converse is given by $P' \longrightarrow P' \cap S$ for all $P' \in Spec(T)$. Furthermore $PT \cap S = P$ for all $P \in Spec_0(S)$ and P is a v-ideal, equivalently P is a projective ideal (left and right projective).

Proof. It follows from the proofs of [9, (2.3.16) and (2.3.17)] that the mapping is a bijection and each $P \in Spec_0(S)$ is a v-ideal. □

Lemma 2.3. *(1) $V_{(m,r)}(S) = \{\mathfrak{m}[x; \delta] : \mathfrak{m}$ is a δ-maximal ideal of $R\} \cup Spec_0(S)$.*
(2) $V_{(m,r)}(S) = V_{(m,l)}(S)$, which is denoted by $V_m(S)$ and an ideal of S is a maximal projective ideal (ideal maximal amongst the projective ideals) if and only if it is a member in $V_m(S)$.
(3) Each $P \in Spec_0(S)$ is invertible.

Proof. (1) Let $A \in V_{(m,r)}(S)$. Assume that $\mathfrak{a} = A \cap R \neq (0)$. Then it is a δ-prime ideal since A is a prime ideal by [8, Lemma 1.1] and so $\mathfrak{a}$ is δ-maximal by Lemma 2.1. To prove $A = \mathfrak{a}[x; \delta]$, we assume, on the contrary, that $A \supset \mathfrak{a}[x; \delta]$, then the leading coefficients ideal

$C(A)$ of A is δ-stable with $C(A) \supset \mathfrak{a}$ and so $C(A) = R$, that is, there is a monic polynomial $a(x) \in A$. To prove that $(S : A)_l = S$, let $q \in (S : A)_l$, that is, $qA \subseteq S$. Then $q \in T$ since $AT = T$. Write $q = q_n x^n + \cdots + q_0$, where $q_i \in Q$. $qa(x) \in S$ implies $q_n \in R$ since $a(x)$ is monic, that is, $q_n x^n a(x) \in S$ and $(q_{n-1}x^{n-1} + \cdots + a_0)a(x) = qa(x) - q_n x^n a(x) \in S$. Thus we inductively have $q_i \in R$ for all i $(0 \leq i \leq n-1)$. Hence $q \in S$ and $(S : A)_l = S$ follows since $(S : A)_l \supseteq S$. Hence $A_v = (S : (S : A)_l)_r = (S : S)_r = S$, which contradicts $A = A_v$. Hence $A = \mathfrak{a}[x;\delta]$.

Conversely, let $\mathfrak{m}$ be a δ-maximal ideal of R and let $A \in V_{(m,r)}$ with $A \supseteq \mathfrak{m}[x;\delta]$. Then $\mathfrak{a} = A \cap R = \mathfrak{m}$. Hence $A = \mathfrak{a}[x;\delta] = \mathfrak{m}[x;\delta]$ as above.

Assume that $A \cap R = (0)$. Then $AT \subset T$ and so there is a maximal ideal P' of T with $P' \supseteq AT$. Thus $P = P' \cap S \supseteq A$ and $P = A$ by Lemma 2.2, that is, $A \in \mathrm{Spec}_0(S)$. Conversely, let $P \in \mathrm{Spec}_0(S)$ and assume that there is a $B \in V_{(m,r)}(S)$ such that $B \supset P$, then $\mathfrak{b} = B \cap R \neq 0$ by Lemma 2.2. It follows as above that $B = \mathfrak{b}[x;\delta]$ and that $S \supset B \supset P \supset (0)$. Then, by [10, (6.3.11) and (6.5.4)]), we have $2 \geq \mathcal{K}(R[x;\delta]) > \mathcal{K}(S/P) > \mathcal{K}(\bar{S}/\bar{B}) = \mathcal{K}(S/B)$, where $\bar{S} = S/P$, which implies $\mathcal{K}(S/B) = 0$, a contradiction, because $S/B = (R/\mathfrak{b})[x;\delta]$ is not Artinian. Hence $P \in V_{(m,r)}(S)$.
(2) By symmetry $V_{(m,l)} = \{\mathfrak{m}[x;\delta]\colon \mathfrak{m}$ is a δ-maximal ideal of $R\,\} \cup \mathrm{Spec}_0(S)$ and so $V_{(m,r)} = V_{(m,l)}$. The last statement follows from [3, Proposition 5.2] since a left (right) projective ideal is a left (right) v-ideal, respectively.
(3) Let $P \in \mathrm{Spec}_0(S)$. If $P = (S : P)_l P$, then $PT = (T : PT)_l PT = T$ by [9, (2.3.16)], a contradiction. Thus $P \subset (S : P)_l P \subseteq S$ and ${}_v((S : P)_l P) = S$. Similarly, $(P(S : P)_r)_v = S$. This implies that P is v-invertible. Hence P is invertible by [1, Lemma 2.2]. □

From Lemmas 2.1 and 2.3 we have the following:

Corollary 2.1. *Any element in $V_m(S)$ is either idempotent or invertible.*

We first describe the structure of invertible ideals of S. A finite set of distinct idempotent maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ $(n > 1)$ is called a *cycle* if $O_r(\mathfrak{m}_1) = O_l(\mathfrak{m}_2), \ldots, O_r(\mathfrak{m}_n) = O_l(\mathfrak{m}_1)$.

Lemma 2.4. *Let P be an ideal of S with $P \cap R \neq (0)$. Then P is a maximal invertible ideal (ideals maximal amongst the invertible ideals) if and only if there is either a cycle $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ or a δ-maximal invertible ideal $\mathfrak{m}^e$ such that either $P = \mathfrak{m}_1[x;\delta] \cap \cdots \cap \mathfrak{m}_n[x;\delta]$ or $P = \mathfrak{m}^e[x;\delta]$.*

Proof. Suppose P is a maximal invertible ideal of S. By Lemmas 2.1 and 2.3, there is either a idempotent maximal ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_1[x;\delta] \supseteq P$ or an invertible maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^e[x;\delta] \supseteq P$. In the latter case it is clear that $P = \mathfrak{m}^e[x;\delta]$. In the first case, as in [1, Lemma 3.7], there is a cycle $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ such that $P = \mathfrak{m}_1[x;\delta] \cap \cdots \cap \mathfrak{m}_n[x;\delta]$. It is also clear that both $P = \mathfrak{m}_1[x;\delta] \cap \cdots \cap \mathfrak{m}_n[x;\delta]$ and $P = \mathfrak{m}^e[x;\delta]$ are maximal invertible ideals. □

Since S is a Noetherian ring, it is clear that any invertible ideal of S is a finite product of maximal invertible ideals and maximal invertible ideals are commute (see the proof of [5, Proposition 2.8.]). Hence we have the following theorem:

Theorem 2.1. *The invertible ideals of S generate an Abelian group and the generators are $\mathfrak{m}_1[x;\delta]\cap\cdots\cap\mathfrak{m}_n[x;\delta]$, where $\mathfrak{m}_1,\ldots,\mathfrak{m}_n$ is a cycle, $\mathfrak{m}^e[x;\delta]$, where $\mathfrak{m}^e$ is a δ-maximal invertible ideal and $P \in Spec_0(S)$.*

Next we describe the structure of ideals A of S such that $A = A_v$ and A is not contained in any invertible ideals.

Lemma 2.5. *Let P be a prime ideal of S with $\mathfrak{p} = P \cap R \neq (0)$. Then either $P = \mathfrak{p}[x;\delta]$ or P contains a monic polynomial and $P_v = S$.*

Proof. It is clear that $\mathfrak{p}$ is a δ-prime ideal and so it is δ-maximal by Lemma 2.3. $P \supseteq \mathfrak{p}[x;\delta]$ implies that $P = \mathfrak{p}[x;\delta]$ or $P \supset \mathfrak{p}[x;\delta]$. In the latter case $C(P) = R$, that is, P contains a monic polynomial and so $P_v = S$ follows (see, the proof of Lemma 2.3). □

Lemma 2.6. *Let A be an ideal of S with $A = A_v$ and there are no invertible ideals containing A. Then $\mathfrak{a} = A \cap R \neq (0)$.*

Proof. Assume that $\mathfrak{a} = (0)$. Then AT is a proper ideal of T. Let P' be a maximal ideal of T which contains AT. Then $P = P' \cap S \supseteq A$ and P is invertible by Lemma 2.3, a contradiction. Hence $\mathfrak{a} \neq (0)$. □

In the remainder of this section, let A be an ideal of S with $A = A_v$ (equivalently, A is a right projective) and there are no invertible ideals containing A. Put $\mathfrak{a} = A \cap R$. We will show that $A = \mathfrak{a}[x;\delta]$ in a similar way as in [1].

Assume that $\{\mathfrak{m}_1[x;\delta],\ldots,\mathfrak{m}_r[x;\delta]\}$ is the full set of maximal projective ideals containing A, that is, $\mathfrak{m}_i$ are idempotent maximal ideals of R (see, Lemmas 2.1, 2.3 and 2.4) and put $\mathfrak{n} = \mathfrak{m}_1 \cap\cdots\cap \mathfrak{m}_r$. Then:

(5) $\mathfrak{n}^s \subseteq C(A)$ for some $s \leq r$ (see [1, Lemma 3.11]).
(6) If $\mathfrak{n}$ is idempotent, then $A = \mathfrak{n}[x;\delta]$ (see, [1, Corollary 3.12]).
(7) Assume that $O_r(\mathfrak{m}_1) = R' = O_l(\mathfrak{m}_2)$ (note that R' is δ-stable) and put $S' = R'[x;\delta]$. Then, for any fractional right S-ideal I and fractional left S-deal J, $(S' : IS')_l = S'(S : I)_l$ and $(S' : S'J)_r = (S : J)_r S'$. In particular, $(IS')_{v'} = I_v S'$ and ${}_{v'}(S'J) = S'({}_vJ)$, where $(IS')_{v'} = (S'(S' : IS')_l)_r$ and ${}_{v'}(S'J) = (S' : (S' : S'J)_r)_l$ ([1, Lemma 3.14]).
(8) Under the same conditions as in (7), $\mathfrak{m}_1 R' = \mathfrak{m}_1, R'\mathfrak{m}_1 = R', R'\mathfrak{m}_2 = \mathfrak{m}_2$ and $\mathfrak{m}_2 R' = R'$, and so $\mathfrak{m}_1[x;\delta]S' = \mathfrak{m}_1[x;\delta], S'\mathfrak{m}_1[x;\delta] = S', S'\mathfrak{m}_2[x;\delta] = \mathfrak{m}_2[x;\delta]$ and $\mathfrak{m}_2[x;\delta]S' = S'$. In particular, R' is a projective R-ideal since it is a fractional R-ideal.

Lemma 2.7. *Under the same conditions as in (7),*
(1) $(S : \mathfrak{m}_2[x;\delta])_l = O_l(\mathfrak{m}_2)[x;\delta] = R'[x;\delta]$ which is S-projective, that is, left and right

S-projective.
(2) $(S : S')_l = \mathfrak{m}_1[x;\delta]$, *which is S-projective.*

Proof. (1) $R'[x;\delta]\mathfrak{m}_2[x;\delta] = R'\mathfrak{m}_2[x;\delta] = \mathfrak{m}_2[x;\delta] \subseteq S$ by (8). Thus $R'[x;\delta] \subseteq (S : \mathfrak{m}_2[x;\delta])_l$. Conversely, let $q \in (S : \mathfrak{m}_2[x;\delta])_l$. Then $q\mathfrak{m}_2[x;\delta] \subseteq S$ implies $q \in qT = q\mathfrak{m}_2[x;\delta]T \subseteq ST = T$. Thus write $q = q_n x^n + \cdots + q_0$, where $q_i \in Q$. Since $q\mathfrak{m}_2 \subseteq S$, we have $q_i \in (R : \mathfrak{m}_2)_l = R'$, that is, $q \in R'[x;\delta]$. Hence $(S : \mathfrak{m}_2[x;\delta])_l = R'[x;\delta]$.
(2) is proved in a similar way as in (1). □

The proof of following proposition is similar to one in [1, Proposition 3.15]. However we give a complete proof of it since the proof here is a shorter and more readable than one in [1].

Proposition 2.1. *Let A be an ideal of S with $A = A_v$ (equivalently A is right projective). If there are no invertible ideals of S containing A, then $A = \mathfrak{a}[x;\delta]$, where $\mathfrak{a} = A \cap R$.*

Proof. $\mathfrak{a} = A \cap R \neq (0)$ by Lemma 2.6. Let $\mathfrak{m}_1[x;\delta], \ldots, \mathfrak{m}_r[x;\delta]$ be the full set of maximal projective idempotent ideals containing A. If either $r = 1$ or $O_r(\mathfrak{m}_i) \neq O_l(\mathfrak{m}_j)$ for all i, j $(1 \leq i, j, \leq r)$, then $A = \mathfrak{a}[x;\delta]$ by (6) and [11, Corollary 5.5].
Thus we may assume that $r \geq 2$ and the assertion is true for any ideal of a differential polynomial ring over HNP ring which is a right projective, there are no invertible ideals containing it and is contained in at most $r - 1$ maximal projective ideals. Furthermore we may also assume, without loss of generality, that $R' = O_r(\mathfrak{m}_1) = O_l(\mathfrak{m}_2)$ and $O_r(\mathfrak{m}_2) \neq O_l(\mathfrak{m}_1)$. Then $\mathfrak{m}_2\mathfrak{m}_1, \mathfrak{m}_3R', \ldots, \mathfrak{m}_rR'$ are all maximal idempotent ideals of R' ([1, Proposition 2.7]). Put $S' = R'[x;\delta]$.
We complete the proof by showing that there are δ-stable ideals $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ and $\mathfrak{b}_4$ of R such that

(i) $(\mathfrak{m}_1[x;\delta]A\mathfrak{m}_2[x;\delta])_v = \mathfrak{b}_1[x;\delta]$,
(ii) $(\mathfrak{m}_1[x;\delta]A\mathfrak{m}_1[x;\delta])_v = \mathfrak{b}_2[x;\delta]$,
(iii) $(\mathfrak{m}_2[x;\delta]A\mathfrak{m}_2[x;\delta])_v = \mathfrak{b}_3[x;\delta]$ and
(iv) $(\mathfrak{m}_2[x;\delta]A\mathfrak{m}_1[x;\delta])_v = \mathfrak{b}_4[x;\delta]$.

The proof of (i): Put $A' = S'AS' = S'\mathfrak{m}_1[x;\delta]A\mathfrak{m}_2S'$. If $S' \supset A'_{v'}$, then we claim that any maximal projective ideals of S' containing $A'_{v'}$ is one of $\mathfrak{m}_2\mathfrak{m}_1[x;\delta], \mathfrak{m}_3S', \ldots, \mathfrak{m}_rS'$. Let $\mathfrak{n}'[x;\delta]$ be a maximal projective ideal containing $A'_{v'}$, where $\mathfrak{n}'$ is a δ-maximal projective ideal of R' and so either $\mathfrak{n}' = \mathfrak{m}_2\mathfrak{m}_1$ or $\mathfrak{n}' = \mathfrak{m}R'$ for some δ-maximal projective ideal $\mathfrak{m}$ of R by [1, Proposition 2.7]. In the latter case, $\mathfrak{n}'[x;\delta] \cap S = (\mathfrak{n}' \cap R)[x;\delta] = \mathfrak{m}[x;\delta] \supseteq A$ and $\mathfrak{m} = \mathfrak{m}_j$ for some j $(1 \leq j \leq r)$.

However, $j \neq 1, 2$ since $S'\mathfrak{m}_1 = S' = \mathfrak{m}_2S'$ by (8). Hence, by induction on r and (7), $A'_{v'} = (A'_v)S' = \mathfrak{b}'[x;\delta]$ for some δ-stable ideal $\mathfrak{b}'$ of R' which is a projective R-ideal. In case $A'_{v'} = S'$, then we put $A'_{v'} = \mathfrak{b}'[x;\delta]$, where $\mathfrak{b}' = R'$.

In both cases, by using (7) and [8, Lemma 1.1], we have

$$\begin{aligned}(\mathfrak{m}_1[x;\delta]A)_v S' &= (\mathfrak{m}_1[x;\delta]AS')_{v'} = (\mathfrak{m}_1[x;\delta]S'AS')_{v'} = (\mathfrak{m}_1[x;\delta]\mathfrak{b}'[x;\delta])_{v'}\\ &= (\mathfrak{m}_1[x;\delta]\mathfrak{b}'[x;\delta])_v S' = (\mathfrak{m}_1[x;\delta]\mathfrak{b}'[x;\delta])S' = \mathfrak{m}_1\mathfrak{b}'[x;\delta]\ ,\end{aligned}$$

since $\mathfrak{m}_1[x;\delta]\mathfrak{b}'[x;\delta]$ is S-projective and a right projective ideal is a right v-ideal. Thus, by (8), [1, Lemma 2.1] and Lemma 2.7 we have

$$\begin{aligned}(\mathfrak{m}_1[x;\delta]A\mathfrak{m}_2[x;\delta])_v &= ((\mathfrak{m}_1[x;\delta]A)_v S'\mathfrak{m}_2[x;\delta])_v\\ &= (\mathfrak{m}_1\mathfrak{b}'[x;\delta]\mathfrak{m}_2[x;\delta])_v = \mathfrak{m}_1\mathfrak{b}'\mathfrak{m}_2[x;\delta] = \mathfrak{b}_1[x;\delta]\ ,\end{aligned}$$

where $\mathfrak{b}_1 = \mathfrak{m}_1\mathfrak{b}'\mathfrak{m}_2$ is a δ-stable ideal of R.

The proof of (ii): Put $I_1 = \mathfrak{m}_1[x;\delta]A\mathfrak{m}_1[x;\delta]$, an (S,S')-ideal. As in (i), $(S'I_1)_{v'}$ is contained in at most $r-1$ maximal projective ideals of S' (one of them is $\mathfrak{m}_2\mathfrak{m}_1[x;\delta]$). By induction hypothesis and (7), $(S'I_1)_v S' = (S'I_1)_{v'} = \mathfrak{b}_2'[x;\delta]$ for some δ-stable ideal $\mathfrak{b}_2'$ of R' which is a projective ideal of R. Hence, by (8), Lemma 2.7 and [1, Lemma 2.1], we have

$$\mathfrak{m}_1\mathfrak{b}_2'[x;\delta] = (\mathfrak{m}_1\mathfrak{b}_2'[x;\delta])_v = (\mathfrak{m}_1(S'I_1)_v S')_v = (\mathfrak{m}_1 I_1 S')_v = (\mathfrak{m}_1[x;\delta]A\mathfrak{m}_1[x;\delta])_v\ ,$$

and $\mathfrak{b}_2 = \mathfrak{m}_1\mathfrak{b}_2'$ is a δ-stable ideal of R.

The proof of (iii): Put $I_2 = (\mathfrak{m}_2[x;\delta]A\mathfrak{m}_2[x;\delta])$, an (S',S)-ideal. As before, $(I_2S')_{v'}$ is contained in at most $r-1$ maximal projective ideals of S'. Thus there is a δ-stable ideal $\mathfrak{b}_3'$ of R' such that $(I_2)_v S' = (I_2S')_{v'} = \mathfrak{b}_3'[x;\delta]$. Hence, by Lemma 2.7, [1, Lemma 2.1] and (8), we have

$$\begin{aligned}\mathfrak{b}_3'\mathfrak{m}_2[x;\delta] &= \mathfrak{b}_3'[x;\delta]\mathfrak{m}_2[x;\delta] = (\mathfrak{b}_3'[x;\delta]\mathfrak{m}_2[x;\delta])_v = ((I_2)_v S'\mathfrak{m}_2[x;\delta])_v\\ &= ((I_2)_v\mathfrak{m}_2[x;\delta])_v = (I_2\mathfrak{m}_2[x;\delta])_v = (\mathfrak{m}_2[x;\delta]A\mathfrak{m}_2[x;\delta])_v\ ,\end{aligned}$$

and $\mathfrak{b}_3 = \mathfrak{b}'_3\mathfrak{m}_2$ is a δ-stable ideal of R.

The proof of (iv): Put $I_3 = \mathfrak{m}_2[x;\delta]A\mathfrak{m}_1[x;\delta]$, an ideal of S'. As before, there is a δ-stable ideal $\mathfrak{b}_4'$ of R' such that $(I_3)_v S' = (I_3S')_{v'} = \mathfrak{b}_4'[x;\delta]$. Hence, by [1, Lemma 2.1] and Lemma 2.7, we have

$$(\mathfrak{m}_2[x;\delta]A\mathfrak{m}_1[x;\delta])_v = (I_3)_v = (I_3S')_v = ((I_3)_v S')_v = (\mathfrak{b}_4'[x;\delta])_v = \mathfrak{b}_4'[x;\delta]\ ,$$

and $\mathfrak{b}_4 = \mathfrak{b}_4'$ is a δ-stable ideal of R. Now, by using (i) ~ (iv), we have

$$\begin{aligned}(\mathfrak{b}_1+\mathfrak{b}_3)[x;\delta] &= ((\mathfrak{b}_1+\mathfrak{b}_3)[x;\delta])_v\\ &= ((\mathfrak{m}_1[x;\delta]A\mathfrak{m}_2[x;\delta])_v + (\mathfrak{m}_2[x;\delta]A\mathfrak{m}_2[x;\delta])_v)_v = (A\mathfrak{m}_2[x;\delta])_v\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{b}_2+\mathfrak{b}_4)[x;\delta] &= ((\mathfrak{b}_2+\mathfrak{b}_4)[x;\delta])_v\\ &= ((\mathfrak{m}_1[x;\delta]A\mathfrak{m}_1[x;\delta])_v + (\mathfrak{m}_2[x;\delta]A\mathfrak{m}_1[x;\delta])_v)_v = (A\mathfrak{m}_1[x;\delta])_v\ ,\end{aligned}$$

since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$. Hence

$$\begin{aligned}(\mathfrak{b}_1 + \mathfrak{b}_2 + \mathfrak{b}_3 + \mathfrak{b}_4)[x;\delta] &= ((A\mathfrak{m}_2[x;\delta])_v + (A\mathfrak{m}_1[x;\delta])_v)_v \\ &= (A\mathfrak{m}_2[x;\delta] + A\mathfrak{m}_1[x;\delta])_v = A_v = A\ ,\end{aligned}$$

which completes the proof. □

Combining Theorem 2.1 with Proposition 2.1, we have the following theorem.

Theorem 2.2. *Let R be an HNP ring with derivation δ and $S = R[x;\delta]$. If A is an ideal of S with $A = A_v$ (equivalently A is right projective), then $A = X\mathfrak{b}[x;\delta]$, where X is an invertible ideal of S and $\mathfrak{b}$ is a δ-stable eventually idempotent ideal of R. In particular, A is left projective as well. Furthermore, if $\{\mathfrak{m}_1[x;\delta], \dots, \mathfrak{m}_r[x;\delta]\}$ is the full set of maximal projective idempotent ideals containing $\mathfrak{b}[x;\delta]$, then $(\mathfrak{b}[x;\delta])^r = \mathfrak{b}^r[x;\delta]$ and is idempotent.*

Proof. We may assume that A is not invertible. Let X be minimal among invertible ideals containing A and put $B = X^{-1}A$. Then $B = B_v$ by [1, Lemma 2.1] and there are no invertible ideals containing B. Thus $B = X^{-1}A = \mathfrak{b}[x;\delta]$ by Proposition 2.1, where $\mathfrak{b} = B \cap R \neq (0)$. Hence $A = X\mathfrak{b}[x;\delta]$. Since ${}_vA = {}_v(X\mathfrak{b}[x;\delta]) = X_v(\mathfrak{b}[x;\delta]) = X\mathfrak{b}[x;\delta] = A$ by the left version of [1, Lemma 2.1], A is left projective. The last statement follows from [5, Proposition 4.5]. □

Note that if A is an ideal of S with $A = {}_vA$, then, by left version of Theorem 2.2, $A = X\mathfrak{b}[x;\delta]$, where X is an invertible ideal and $\mathfrak{b}$ is a δ-stable eventually idempotent ideal of R, and so A is a projective ideal.

In [2], we defined the concept of generalized hereditary Noetherian prime rings (G-HNP rings for short) to study the structure of projective ideals as follows: A prime Goldie ring R is called a *G-HNP ring* if

(i) Any ideal A of R with $A = {}_vA$ (or $A = A_v$) is a projective ideal.
(ii) R is τ-Noetherian (see [9] for the definition of τ-Noetherian).

A G-HNP ring is called a *strongly G-HNP* ring if any one sided v-ideal is projective.
From Theorem 2.2, its note and [3, proposition 5.2], we have the following corollary:

Corollary 2.2. $S = R[x;\delta]$ *is a strongly G-HNP ring.*

We end this section with the following remark:

Remark 2.1. *Noetherian prime rings with global dimension two are not necessarily to be G-HNP rings (see [1]).*

3 Examples

In this section, we will give some examples of HNP rings R with derivations δ to explain various phenomena about projective ideals of $S = R[x;\delta]$.

Let $Q = M_n(D)$ be a matrix ring over the division ring D $(n \geq 2)$, $e \in Q$ be idempotent different from 1, $R' = Q[y]$, the polynomial ring over Q indeterminate y, $\mathfrak{a} = eQ + yR'$, a right ideal of R' with $R'\mathfrak{a} = R'$. Put

$$R = \mathbf{I}_{R'}(\mathfrak{a}) = \{r' \in R' \mid r'\mathfrak{a} \subseteq \mathfrak{a}\},$$

the *idealizer* of $\mathfrak{a}$ in R'.

Put $\mathfrak{b} = Q(1-e) + yR'$, a left ideal of R' with $\mathfrak{b}R' = R'$.

Proposition 3.1. *(1) R is an HNP ring and $R = Q(1-e) + eQ + yR' = Q(1-e) + \mathfrak{a}$. Furthermore,*

$$R = \mathbf{I}_{R'}({}_{R'}\mathfrak{b}) = \{r' \in R' \mid \mathfrak{b}r' \subseteq \mathfrak{b}\}\,.$$

(2) $\mathfrak{a}$ and $\mathfrak{b}$ are both idempotent maximal ideals of R and $\mathfrak{a}, \mathfrak{b}$ is a cycle. In particular, $\mathfrak{a} \cap \mathfrak{b}$ is a maximal invertible ideal of R.

Proof. (1) Since $\mathfrak{a}$ is a generative and isomaximal right ideal of R' in the sense of [10], R is an HNP ring by [10, (5.5.10)]. It is clear that

$$R \supseteq Q(1-e) + eQ + yR'\ . \qquad (*)$$

Conversely let $r \in R$. Then $r = q_0 + yr' = q_0(1-e) + q_0e + yr'$, where $q_0 \in Q$ and $r' \in R'$. By (*), $q_0e \in R$ and so $q_0e = q_0ee \subseteq q_0e\mathfrak{a} \subseteq \mathfrak{a} = eQ + yR'$, which implies $q_0e \in eQ$. Hence $r \in Q(1-e) + eQ + yR'$. By the left hand version, $R = eQ + \mathfrak{b} = eQ + Q(1-e) + yR'$.

(2) It follows from [10, (5.5.10)] that $\mathfrak{a}$ and $\mathfrak{b}$ are idempotent maximal ideals such that $O_r(\mathfrak{a}) = R' = O_l(\mathfrak{b})$. To prove $\mathfrak{a}, \mathfrak{b}$ is a cycle, we prove that $O_l(\mathfrak{a}) = eQ(1-e)y^{-1} + R$. Clearly $O_l(\mathfrak{a}) \supseteq eQ(1-e)y^{-1} + R$. Let $q \in O_l(\mathfrak{a})$, that is, $q\mathfrak{a} \subseteq \mathfrak{a}$ and so $qyR' \subseteq \mathfrak{a}R' \subseteq R'$.Thus $q \in R'y^{-1}$. Write $q = g(y)y^{-1}$, where $g(y) = q_0 + q_1y + \cdots + q_ly^l \in R'$ and $q = q_0y^{-1} + q_1 + (q_2y + \cdots + q_ly^{l-1})$, which implies $q_0y^{-1} + q_1 \in O_l(\mathfrak{a})$.

So, in particular, $(q_0y^{-1} + q_1)e \in eQ + yR'$ and $(q_0y^{-1} + q_1)y \in eQ + yR'$. Thus $q_0e = 0$, $q_0 \in eQ$ and $q_1e \in eQ$. It follows that $q_0 \in eQ(1-e)$ and that $q_1 = q_1e + q_1(1-e) \in eQ + Q(1-e) \subseteq R$. Thus $q \in eQ(1-e)y^{-1} + R$ and hence $O_l(\mathfrak{a}) = eQ(1-e)y^{-1} + R$. Similarly $O_r(\mathfrak{b}) = eQ(1-e)y^{-1} + R$. Hence $\mathfrak{a}, \mathfrak{b}$ is a cycle and $\mathfrak{a} \cap \mathfrak{b}$ is a maximal invertible ideal of R by [5, Theorem 2.6]. □

Lemma 3.1. *Let $f(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_0$ be a monic polynomial in R'. Then $f(y)R'$ is an ideal if and only if $f(y) \in K[y]$, where K is the center of Q. In particular $f(y)R'$ is a prime ideal if and only if it is an irreducible polynomial in $K[y]$.*

Proof. By ([4] or [9, p135]), $f(y)R'$ is an ideal of R' if and only if $f(y)a = af(y)$ for all $a \in Q$ and $f(y)y = (y + b)f(y)$ for some $b \in Q$. It follows that $a_i \in K$ for all i and $b = 0$, that is, equivalently, $f(y) \in K[y]$. The last statement is clear. □

In the remainder of this section we consider the following derivation δ of R' (see, [6, Lemma 1.2]):

$$\delta(y) = y^2 \text{ and } \delta(a) = 0 \text{ for all } a \in Q .$$

Lemma 3.2. *Let $f(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_0 \in K[y]$. Then for any natural number l and for $a \in Q$, $\delta(ay^l) = lay^{l+1}$ and $\delta(f(y)^l) = lf(y)^{l-1}\delta(f(y)) = lf(y)^{l-1}h(y)y^2$, where $h(y) = ny^{n-1} + (n-1)a_{n-1}y^{n-2} + \cdots + a_1$.*

Proof. It is easily proved by induction on l. □

Note that $R' = Q[y]$ is a principal ideal ring.

Lemma 3.3. *Let $\mathfrak{p}'$ be a prime ideal of R', where $\mathfrak{p}' = f(y)R'$ and $f(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_0$ is an irreducible polynomial in $K[y]$.*
(1) In case $a_0 = 0$. $\mathfrak{p}' = yR'$ and $\mathfrak{p}'$ is δ-stable.
(2) In case $a_0 \neq 0$.
(i) If char $K = 0$, then $\mathfrak{p}'^l$ is not δ-stable for all $l \geq 1$.
(ii) If char $K = p \neq 0$, then
(a) $\mathfrak{p}'$ is δ-stable if and only if $\delta(f(y)) = 0$.
(b) $\mathfrak{p}'^p$ is always δ-stable. If $\mathfrak{p}'$ is not δ-stable, then $\mathfrak{p}'^l$ is not δ-stable for any $l(1 \leq l < p)$. In particular, if $\deg f(y) = 1$, then $\mathfrak{p}'$ is not δ-stable.

Proof. (1) is clear.
(2) (i) If $\mathfrak{p}'^l$ is δ-stable, then $f(y)^l R' \ni \delta(f(y)^l) = lf(y)^{l-1}h(y)y^2$ by Lemma 3.2 and so $\mathfrak{p}' \ni h(y)y^2$, which implies either $\mathfrak{p}' \ni y^2$ or $\mathfrak{p}' \ni h(y)$. In the first case, $\mathfrak{p}' = yR'$ which does not happen since $a_0 \neq 0$. The second case is also impossible since $\deg h(y) = n-1$ and $\deg f(y) = n$.
(ii) (a) If $\mathfrak{p}'$ is δ-stable, then as in (2) (i), we have $\mathfrak{p}' \ni h(y)$ and so $h(y) = 0$ since deg $h(y) \leq n-1$ and $f(y)$ is a monic polynomial with $\deg f(y) = n$. Hence $\delta(f(y)) = 0$ by Lemma 3.2. Of course, $\mathfrak{p}'$ is δ-stable if $\delta(f(y)) = 0$.
(b) $\mathfrak{p}'^p$ is δ-stable by Lemma 3.2. If $\mathfrak{p}'$ is not δ-stable and $\mathfrak{p}'^l$ is δ-stable for some $l(1 \leq l < p)$, then, as before, we have $f(y)R' \ni h(y)y^2$ which is again impossible. The last statement is clear.
Note that δ induces a derivation of R and we use [10, (5.6.11)] to study the structure of prime ideals of R. □

Lemma 3.4. *(1) Spec(R) $= \{\mathfrak{a}, \mathfrak{b}, \mathfrak{p} = \mathfrak{p}' \cap R \mid \mathfrak{p}' = f(y)R' \in$ Spec (R') with $f(y) \neq y\}$ and $\mathfrak{p}$ is an invertible ideal of R.*
(2) $\mathfrak{p}'^l = \mathfrak{p}^l R'$ and $\mathfrak{p}^l = \mathfrak{p}'^l \cap R$ for any $l \geq 1$.
(3) $\mathfrak{p}$ is δ-stable if and only if so is $\mathfrak{p}'$.

Proof. (1) follows from [10, (5.6.11)] (note that $\mathfrak{a} \supset yR'$).
(2) Since $\mathfrak{b}$ is a left R'-ideal with $\mathfrak{b}R' = R'$, we have $\mathfrak{p}'\mathfrak{b} \subseteq \mathfrak{b}$ and so $\mathfrak{p}'\mathfrak{b} \subseteq R \cap \mathfrak{p}' = \mathfrak{p}$. Thus $\mathfrak{p}' = \mathfrak{p}'R' = \mathfrak{p}'\mathfrak{b}R' \subseteq \mathfrak{p}R'$. Hence $\mathfrak{p}' = \mathfrak{p}R'$ follows. Suppose $\mathfrak{p}'^{l} = \mathfrak{p}^{l}R'$ inductively. Then $\mathfrak{p}'^{(l+1)} = \mathfrak{p}'^{l}\mathfrak{p}' = \mathfrak{p}^{l}R'\mathfrak{p}R' = \mathfrak{p}^{l}\mathfrak{p}' = \mathfrak{p}^{(l+1)}R'$ and so $\mathfrak{p}'^{l} = \mathfrak{p}^{l}R'$ for any $l \geq 1$. To prove $\mathfrak{p}^{l} = \mathfrak{p}'^{l} \cap R$ for any $l \geq 1$, we may assume that $\mathfrak{p}^{l} = \mathfrak{p}'^{l} \cap R$ for some l. If $\mathfrak{p}^{l+1} \subset \mathfrak{p}'^{l+1} \cap R$, then $\mathfrak{p}^{l+1} \subset \mathfrak{p}'^{l+1} \cap R \subseteq \mathfrak{p}'^{l} \cap R = \mathfrak{p}^{l}$ and so $\mathfrak{p} \subset (\mathfrak{p}'^{l} \cap R)\mathfrak{p}^{-l} \subseteq R$. Thus $(\mathfrak{p}'^{l+1} \cap R)\mathfrak{p}^{-l} = R$ since $\mathfrak{p}$ is a maximal ideal of R and $\mathfrak{p}'^{l+1} \cap R = \mathfrak{p}^{l}$ follows. Then $\mathfrak{p}'^{l+1} \supseteq \mathfrak{p}^{l}R' = \mathfrak{p}'^{l}$, a contradiction. Hence $\mathfrak{p}^{l} = \mathfrak{p}'^{l} \cap R$ for any $l \geq 1$.
(3) follows from (2). □

Combining Lemmas 3.3 with Lemma 3.4, we have the following proposition:

Proposition 3.2. *Under the same notation as in Lemma 3.4.*
(1) If char $K = 0$, then there are no δ-prime ideals of R which is invertible. $\mathfrak{a}$ and $\mathfrak{b}$ are only idempotent maximal ideals of R.
(2) If char $K = p \neq 0$, then $\mathfrak{p}$ is either a δ-prime ideal or $\mathfrak{p}^{l}$ is not δ-stable for any l $(1 \leq l < p)$and $\mathfrak{p}^{p}$ is a δ-prime ideal.

In the reminder of this section, let $S = R[x;\delta]$ and $T = Q(R)[x;\delta]$, where $Q(R)$ is the quotient ring of R, that is, $Q(R) = Q(R') = Q(y)$.

Lemma 3.5. *If char $K = 0$. Then T is a simple ring.*

Proof. If T is not a simple ring, then, by [7, Proposition 2.8], there is a monic semi-invariant polynomial $p(x) = x + a$, where $a \in Q(R)$, that is, for each $b \in Q(R)$ there is $c \in Q(R)$ such that $p(x)b = cp(x)$. It follows that $bx + \delta(b) + ab = cx + ca$. Thus $b = c$ and $\delta(b) = ba - ab$, an inner derivation induced by a. In case $b = y$, $\delta(y) = 0$, a contradiction. Hence T is a simple ring. □

Lemma 3.6. *Suppose char $K = p \neq 0$. Then*
(1) $(x^{p} + 1)T$ is the unique maximal ideal of T and $(x^{p} + 1)$ is a central element.
(2) $(x^{p} + 1)S \in Spec_{0}(S)$.

Proof. For any $r' = \sum_{i=0}^{l} a_i y^i \in R'$, we claim that $(x^p+1)r' = r'(x^p+1)$. $(x^p+1)r' = x^p r' + r' = \sum_{i=0}^{p} {}_pC_i\delta^i(r')x^{p-i} + r' = r'x^p + \delta^p(r') + r'$. Since δ is additive and $\delta(ay^i) = a\delta(y^i)$ for any $a \in Q$ and any i, we have $\delta^p(r') = \sum_{i=0}^{l} a_i\delta^p(y^i)$ and $\delta^p(y^i) = i(i+1)\dots(i+p-1)y^{i+p} = 0$. Hence $(x^p+1)r' = r'(x^p+1)$ and x^p+1 is a central element since $(x^p+1)x = x(x^p+1)$. By [7, Proposition 2.8] and [4, Theorem 5.1.4], $(x^p+1)T$ is the unique maximal ideal of T.
(2) It is clear that $(x^p+1)T \cap S \supseteq (x^p+1)S$. Let $(x^p+1)s' \in (x^p+1)T \cap S$, where $s' = \sum_{i=0}^{l} q_i x^i \in T$. If $l = 0$, then $S \ni (x^p+1)q_0 = q_0x^p +$ (the lower degree parts) and so $q_0 \in R$. Thus $(x^p+1)q_0 \in (x^p+1)S$. If $l \geq 1$, then $S \ni (x^p+1)s' = (x^p+1)q_lx^l + (x^p+1)(\sum_{i=0}^{l-1} q_ix^i) = q_lx^{p+l} +$ (the lower degree parts). Thus $q_l \in R$ and $(x^p+1)q_lx^l \in (x^p+1)S \subseteq (x^p+1)T \cap S$. Since $(x^p+1)(\sum_{i=0}^{l-1} q_ix^{-i}) = (x^p+1)s' - (x^p+1)q_lx^l \in (x^p+1)T \cap S$,

by induction on l, $(x^p+1)(\sum_{i=0}^{l-1} q_i x^i) \in (x^p+1)S$ and thus $(x^p+1)s' \in (x^p+1)S$. Hence $(x^p+1)T \cap S = (x^p+1)S$. By Lemma 2.2, $(x^p+1)S \in Spec_0(S)$. □

Finally we give the following remarks which show various phenomena about projective ideals in $S = R[x;\delta]$.

Remark 3.1. *(1) In case char* $K = 0$.
(i) $Spec_0(S) = \phi$ *(Lemmas 2.2 and 3.5).*
(ii) Maximal projective ideals of S *are only* $\mathfrak{a}[x;\delta]$ *and* $\mathfrak{b}[x;\delta]$, *and* $(\mathfrak{a} \cap \mathfrak{b})[x;\delta]$ *is the unique maximal invertible ideal (Lemmas 2.3, 2.4, 3.4 and Propositions 3.1, 3.2).*
(2) In case char $K = p \neq 0$.
(i) $Spec_0(S) = \{(x^p+1)S\}$ *(Lemmas 2.2 and 3.6]).*
(ii) $\{\mathfrak{a}[x;\delta], \mathfrak{b}[x;\delta], \mathfrak{p}[x;\delta], \mathfrak{p}^p[x;\delta], (x^p+1)S \mid \mathfrak{p}$ *is a* δ*-stable invertible maximal ideal of* R, $\mathfrak{p}$ *is not* δ*-stable and* $\mathfrak{p}^p$ *is a* δ*-maximal invertible ideal*$\}$ *is the set of all maximal projective ideals of* S. *Furthermore* $\{(\mathfrak{a} \cap \mathfrak{b})[x;\delta], \mathfrak{p}[x;\delta], \mathfrak{p}^p[x;\delta], (x^p+1)S \mid \mathfrak{p}$ *is a* δ*-stable invertible maximal ideal of* R, *and* $\mathfrak{p}$ *is not* δ*-stable,* $\mathfrak{p}^p$ *is a* δ*-maximal invertible ideal*$\}$ *is the set of all maximal invertible ideals of* S *(Lemmas 2.1, 2.3, 3.4 and Proposition 3.2).*

Remark 3.2. *(1) Suppose* K *is an imperfect field. If char* $K = 2$, *then* $f(y) = y^2 + a_0$ *is irreducible if and only if* $\sqrt{a_0} \notin K$. *Similarly if char* $K = 3$, *then* $f(y) = y^3 - a_0$ *is irreducible if and only if* $\sqrt[3]{a_0}$ *is not in* K. *In both cases* $\mathfrak{p}' = f(y)R'$ *is* δ*-stable so that* $\mathfrak{p}[x;\delta]$ *is a maximal invertible ideal of* S.
(2) If char $K = p \neq 0$ *and* $f(y) = y + a_0$ $(a_0 \neq 0)$, *then* $\mathfrak{p}' = f(y)R'$ *is not* δ*-stable (Lemma 3.3). So* $\mathfrak{p}$ *is not* δ*-stable and* $\mathfrak{p}^p[x;\delta]$ *is a maximal invertible ideal of* S. *In the case of a finite field* K, *there are so many examples of irreducible polynomials* $f(y) \in K[y]$ *such that* $\mathfrak{p}' = f(y)R'$ *is not* δ*-stable, which implies* $\mathfrak{p}^p[x;\delta]$ *is a maximal invertible ideal of* S.

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Tsiu-Kwen Lee

Additive central m-power skew-commuting maps on semiprime rings

Abstract: Let R be an $m!$-torsion free semiprime ring with maximal left ring of quotients $Q_{ml}(R)$, where m is a positive integer greater than 1. Given an additive map $f\colon R \to Q_{ml}(R)$, it is proved that $f(x)x^m + x^m f(x) \in C$ for all $x \in R$ if and only if $f(x)R \subseteq C$ and $Rf(x) \subseteq C$ for all $x \in R$, where C denotes the extended centroid of R. In particular, additive m-power skew-commuting maps of semiprime rings are characterized under some assumptions.

Keywords: Central m-power skew-commuting map; extended centroid; maximal left ring of quotients; prime ring; semiprime ring.

1 Results

Rings in the paper are always associative but are not necessarily with unity. A ring R is called semiprime (resp. prime) if, for $a \in R$, $aRa = 0$ implies $a = 0$ (resp. for $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$). Throughout, let R be always a semiprime ring with maximal left ring of quotients $Q_{ml}(R)$. The ring $Q_{ml}(R)$ is also a semiprime ring and its center, denoted by C, is called the extended centroid of R. It is known that C is a commutative regular self-injective ring. Moreover, R is a prime ring if and only if C is a field. We refer the reader to the book [2] for details. For $a, b \in R$ we write $[a, b] := ab - ba$, the commutator of a and b. Given additive subgroups A, B of R, $[A, B]$ (resp. AB) is the additive subgroup of R generated by all elements $[a, b]$ (resp. ab) for $a \in A$ and $b \in B$. If $B = \mathbb{Z}b$ for some $b \in R$, we write $[A, b]$ (resp. Ab) to stand for $[A, B]$ (resp. AB) for brevity. We let $\mathrm{M}_n(S)$ stand for the n by n matrix ring over the ring S.

Given a positive integer m, a map $f\colon R \to R$ is called an m-power commuting map (resp. an m-power skew-commuting map) if $[f(x), x^m] = 0$ (resp. $f(x)x^m + x^m f(x) = 0$) for all $x \in R$. A 1-power commuting map (resp. 1-power skew-commuting map) is called a commuting map (resp. a skew-commuting map) for brevity. Brešar proved that every additive commuting map of a prime ring R is of the form $x \mapsto \lambda x + \mu(x)$ for $x \in R$, where $\lambda \in C$ and $\mu\colon R \to C$ (see [4, Theorem 3.2]). Also, see [6, Corollary 4.2] and [16, Theorem 3] for the semiprime case. In [5] Brešar proved that every additive skew-commuting map on a 2-torsion free semiprime ring must be zero. The goal of the paper is to characterize additive central m-power skew-commuting maps from a

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ring R to $Q_{ml}(R)$ if R is an $m!$-torsion free semiprime ring. To be precise, we prove the following main result.

Theorem 1.1. *Let R be an $m!$-torsion free semiprime ring, where m is a positive integer greater than 1. Suppose that $f\colon R \to Q_{ml}(R)$ is an additive map satisfying $f(x)x^m + x^m f(x) \in C$ for all $x \in R$. Then $f(x)R \subseteq C$ and $Rf(x) \subseteq C$ for all $x \in R$.*

We remark that Theorem 1.1 is false if $m = 1$. For instance, let $R = \mathrm{M}_2(C)$, where C is a field, and let $f\colon R \to R$ be the additive map defined by $f(x) = x - \mathrm{tr}(x)$ for all $x \in R$, where $\mathrm{tr}(x)$ is the trace of x. Since $x^2 - \mathrm{tr}(x)x \in C$ for all $x \in R$, we see that $f(x)x + xf(x) \in C$ for all $x \in R$. Clearly, given an $x \in R$, $f(x)R \nsubseteq C$ in general. In fact, the map $x \mapsto x - \mathrm{tr}(x)$ for $x \in R$ is the only counterexample in some sense (see Theorems 2.2 and 2.3 below).

A. Fošner and Rehman posed a conjecture: Let R be a semiprime ring and m be a positive integer greater than 1. Then, with some torsion free assumption on R, any additive m-power skew-commuting map $f\colon R \to R$ is zero (see [8]). M. Fošner proved the above conjecture for $m = 2$ (see [9]). In a recent paper M. Fošner et al. answered the conjecture in the affirmative when the considered semiprime ring R is $m!$-torsion free (see [10, Theorem 4]). However, the proof of [10, Theorem 4] contains a gap. Indeed, applying [10, Theorem 2] to solve the identity "$\overline{f}(\overline{x})\overline{x}^n + \overline{x}^n\overline{f}(\overline{x}) = 0$ for all $\overline{x} \in R/P_\alpha$", they have to require that $\mathrm{char}(R/P_\alpha) = 0$ or a prime $p > n$. However, they only choose prime ideals P_α satisfying $\mathrm{char}(R/P_\alpha) \neq 2$. We also note that the same situation occurs in the proof of [8, Main Theorem, p.208].

As a consequence of Theorem 1.1 we have the following.

Theorem 1.2. *Let R be an $m!$-torsion free semiprime ring, where m is a positive integer greater than 1. Then any additive m-power skew commuting map from R to $Q_{ml}(R)$ is zero.*

Let $Z(R)$ denote the center of the ring R. An additive map $f\colon R \to Q_{ml}(R)$ is called $Z(R)$-linear if $f(\beta x) = \beta f(x)$ for all $\beta \in Z(R)$ and $x \in R$. In particular, any additive map $f\colon R \to Q_{ml}(R)$ is $Z(R)$-linear if $Z(R) = 0$. Replacing the $m!$-torsion free assumption on the considered semiprime ring R by the $Z(R)$-linearity on the considered additive map, we have the following.

Theorem 1.3. *Let R be a 2-torsion free semiprime ring. Suppose that $f\colon R \to Q_{ml}(R)$ is a $Z(R)$-linear map. Suppose that $f(x)x^m + x^m f(x) = 0$ for all $x \in R$, where m is a positive integer. Then $f = 0$.*

Motivated by Theorems 1.2 and 1.3, it seems reasonable to raise the following.

Question 1.4: Let R be a 2-torsion free semiprime ring. Is any additive m-power skew-commuting map from R to $Q_{ml}(R)$ zero?

2 The prime case

Throughout this section, R always denotes a prime ring with extended centroid C. The first aim is to prove the prime case of Theorem 1.1.

Theorem 2.1. *Let R be a noncommutative prime ring with* char $(R) = 0$ *or a prime $p > m$, where m is a positive integer greater than* 1. *Suppose that $f\colon R \to Q_{ml}(R)$ is an additive map satisfying $f(x)x^m + x^m f(x) \in C$ for all $x \in R$. Then $f = 0$.*

To begin with, we need the following.

Lemma 2.1. *Let R be a noncommutative prime ring, $a \in R$, and m a positive integer. Suppose that $[x, xax^m + x^{m+1}a] = 0$ for all $x \in R$. Then $a \in Z(R)$.*

Proof. By assumption,

$$[x, xax^m + x^{m+1}a] = 0 \tag{1}$$

for all $x \in R$. In view of [2, Theorem 6.4.1] or [7, Theorem 2], (1) holds for all $x \in RC$. Suppose on the contrary that $a \notin C$. Then (1) implies that R is a GPI-ring (see [18] or [7] for the definition of prime GPI-rings). In view of Martindale's theorem (see [18, Theorem 3]), RC is a primitive ring with a minimal idempotent $e \in RC$ such that $eRCe$ is a finite-dimensional central division C-algebra. Thus, RC is a dense subring of $\mathrm{End}(V_D)$, where $V = RCe$ and $D = eRCe$.
Case 1: $\dim_D V > 1$. Suppose on the contrary that there exists a vector $v \in V$ such that av and v are linearly independent over D. Choose an element $x \in RC$ such that $xav = av \neq 0$ and $xv = 0$. Then

$$0 = \left[x, xax^m + x^{m+1}a\right]v = x(xax^m + x^{m+1}a)v = av \neq 0\,,$$

a contradiction. This proves that, for any $v \in V$, av and v are linearly dependent over D. Since $\dim_D V > 1$, there exists $\beta \in D$ such that $av = v\beta$ for all $v \in V$. We claim that $\beta \in C$. Indeed, let $d \in D$ and $v \in V$ be arbitrary. Then $a(vd) = (vd)\beta = v(d\beta)$. On the other hand, $a(vd) = (av)d = (v\beta)d = v(\beta d)$. This implies that $v(d\beta - \beta d) = 0$. Since $v \in V$ is arbitrary, $d\beta = \beta d$ for all $d \in D$. Thus, $\beta \in C$. So $a = \beta \in C$. Since $a \in R$, we get $a \in Z(R)$.
Case 2: $\dim_D V = 1$. In this case, RC is a finite-dimensional central division C-algebra. Since R is not commutative, by Wedderburn's theorem C is an infinite field. Choose a maximal subfield F of RC. Then $RC \otimes_C F \cong \mathrm{M}_n(F)$, where $\dim_C RC = n^2 > 1$. Moreover, by a standard argument, $\mathrm{M}_n(F)$ also satisfies (1). Let $e = e^2 \in \mathrm{M}_n(F)$. Replacing x by e in (1), we get $[e, eae + ea] = 0$, implying $ea(1 - e) = 0$. Replacing e by the idempotent $1 - e$, we get $(1 - e)ae = 0$. Thus, $[e, a] = 0$. Since $\mathrm{M}_n(F)$ is spanned by idempotents as a vector space over F, this implies $a \in F$. But $a \in R$, implying $a \in Z(R)$. ☐

Let R be a prime PI-ring. Then RC is a finite-dimensional central simple C-algebra (see [19, Corollary 1]). There exists an extension field F of C such that $RC \otimes_C F \cong \mathrm{M}_n(F)$,

where $\dim_C RC = n^2$. We let $\mathrm{tr}(x)$ (resp. $\det(x)$) be the trace (resp. norm) of $x \in \mathrm{M}_n(F)$. We consider RC as a C-subalgebra of $\mathrm{M}_n(F)$. It is known that if $x \in RC$, then $\mathrm{tr}(x)$ is independent of the choice of the extension field F and $\mathrm{tr}(x) \in C$. For $x \in RC$, we call $\mathrm{tr}(x)$ (resp. $\det(x)$) the reduced trace (resp. reduced norm) of x, denoted by $\mathrm{Rtr}(x)$ (resp. $\mathrm{Rn}(x)$).

Lemma 2.2. *Let R be a noncommutative prime PI-ring,* $\mathrm{char}\,(R) \neq 2$. *Suppose that $(\lambda x + \mu(x))x^m \in C$ for all $x \in R$, where $\lambda \in C$, $\mu\colon R \to C$ is a $Z(R)$-linear map and m is a positive integer.*
Case 1: $\lambda = 0$. Then $\mu = 0$.
Case 2: $\lambda \neq 0$. Then $RC \otimes_C F \cong \mathrm{M}_2(F)$ for some extension field F of C, and $\det(x)x^{m-1} \in F$ *for all $x \in \mathrm{M}_2(F)$. In particular,* $\mathrm{Rn}(x)x^{m-1} \in C$ *and* $\mu(x) = -\lambda\,\mathrm{Rtr}(x)$ *for all $x \in R$.*

Proof. Since R is a prime PI-ring and $\mu\colon R \to C$ is a $Z(R)$-linear map, it is known that there exist finitely many $c_j, d_j \in RC$ such that $\mu(x) = \sum_j c_j x d_j$ for all $x \in R$ (see, for instance, [17, Lemma 2.2]). By assumption, we have

$$\lambda x^{m+1} + \Big(\sum_j c_j x d_j\Big)x^m \in C \tag{2}$$

for all $x \in R$. Note that (2) also holds for all $x \in RC$ (see [2, Theorem 6.4.1] or [7, Theorem 2]). If C is a finite field, then $RC \cong \mathrm{M}_n(C)$, where $\dim_C RC = n^2$. If C is an infinite field, then $RC \otimes_C \overline{C} \cong \mathrm{M}_n(\overline{C})$, where $\overline{C}$ denotes the algebraic closure of C. We let F be the field C if C is a finite field and be the field $\overline{C}$ if C is an infinite field. By (2), applying a standard argument we get

$$\lambda x^{m+1} + \Big(\sum_j c_j x d_j\Big)x^m \in F \tag{3}$$

for all $x \in \mathrm{M}_n(F)$. Note that μ can be uniquely extended to an F-linear map, denoted by the same μ also, from $\mathrm{M}_n(F)$ to F. Moreover, $\mu(x) = \sum_j c_j x d_j$ for all $x \in \mathrm{M}_n(F)$. Thus, by (3) we get

$$\lambda x^{m+1} + \mu(x)x^m \in F \tag{4}$$

for all $x \in M_n(F)$. Suppose that $\lambda = 0$. Then $\mu(x)x^m \in F$ for all $x \in \mathrm{M}_n(F)$. Let $e \in \mathrm{M}_n(F)$ be a nontrivial idempotent. Then $\mu(e)e \in F$, implying $\mu(e) = 0$. Since $\mathrm{M}_n(F)$ is generated by idempotents as a vector space over F and μ is F-linear, we get $\mu = 0$, as asserted.

Suppose next that $\lambda \neq 0$. By (4), we have

$$x^{m+1} + \mu_1(x)x^m \in F \tag{5}$$

for all $x \in M_n(F)$, where $\mu_1 := \lambda^{-1}\mu$. Let e be a nontrivial idempotent in $\mathrm{M}_n(F)$. By (5), $(\mu_1(e) + 1)e \in F$ and so $\mu_1(e) = -1$. Let $x \in R$. Then $e + ex(1-e)$ and $e + (1-e)xe$ are also nontrivial idempotents. Thus $\mu_1(e + ex(1-e)) = -1 = \mu(e + (1-e)xe)$. So $\mu_1([e, x]) = 0$.

Let E denote the additive subgroup generated by all idempotents in $\mathrm{M}_n(F)$. Then $\mu_1([E, \mathrm{M}_n(F)]) = 0$. It is known that E is a noncentral Lie ideal of $\mathrm{M}_n(F)$. Given a nontrivial idempotent $e \in M_n(F)$, $ex(1-e) = [e, ex(1-e)] \in [E, \mathrm{M}_n(F)]$ for all $x \in \mathrm{M}_n(F)$. Clearly, $e\mathrm{M}_n(F)(1-e) \nsubseteq F$. Thus, $[E, \mathrm{M}_n(F)]$ is a noncentral Lie ideal of $\mathrm{M}_n(F)$. In view of Herstein's theorem (see [11, Theorem 1.5]), $[\mathrm{M}_n(F), \mathrm{M}_n(F)] \subseteq [E, \mathrm{M}_n(F)]$ as $\mathrm{char}\,(F) \neq 2$. Up to now, we have proved that $\mu_1([\mathrm{M}_n(F), \mathrm{M}_n(F)]) = 0$.

Let $x, y \in \mathrm{M}_n(F)$. Then $\mu_1([x,y]) = 0$. It follows from (5) that $[x,y]^{m+1} \in F$. In particular, $[e_{12}, e_{21}]^{m+1} = e_{11} + (-1)^{m+1} e_{22} \in F$. Thus, $n = 2$ follows. Let $x \in \mathrm{M}_2(F)$. Then $\mu_1(1) = \mu_1(e_{11} + e_{22}) = -2$, $\mathrm{tr}(2x - \mathrm{tr}(x)) = 0$ and so

$$0 = \mu_1(2x - \mathrm{tr}(x)) = 2\mu_1(x) - \mathrm{tr}(x)\mu_1(1) = 2\Big(\mu_1(x) + \mathrm{tr}(x)\Big)$$

and so $\mu_1(x) = -\mathrm{tr}(x)$. In particular, if $x \in R$ then $\mu_1(x) = -\mathrm{Rtr}(x)$ and $\det(x) = \mathrm{Rn}(x)$. Then

$$x^{m+1} + \mu_1(x)x^m = x^{m-1}\Big(x^2 + \mu_1(x)x\Big) = x^{m-1}\Big(x^2 - \mathrm{tr}(x)x\Big) = x^{m-1}\det(x) \in F$$

for all $x \in \mathrm{M}_2(F)$. Thus, for $x \in R$, we have $x^{m-1}\mathrm{Rn}(x) \in C$ and $\mu(x) = -\lambda\,\mathrm{Rtr}(x)$ for all $x \in R$. ☐

Recall that C is a field. For $x \in R$, $\deg(x)$ is defined as the minimal algebraic degree over C if x is algebraic over C and $\deg(x) = \infty$, otherwise. We let $\deg(R) := \sup\{\deg(t) \mid t \in R\}$. It is known that, for a positive integer m, $\deg(R) > m$ if and only if $\dim_C RC > m^2$. In particular, if R is not a PI-ring, then $\deg(R) = \infty$. Given a map $g \colon R^{n-1} \to Q_{ml}(R)$ and $1 \le i \le n$, we let

$$g(\overline{x}_n^i) = g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

for $\overline{x}_n := (x_1, x_2, \ldots, x_n) \in R^n$.

Lemma 2.3. *Let R be a noncommutative prime ring,* $\mathrm{char}\,(R) = 0$ *or a prime $p > m$, where m is a positive integer greater than* 1. *Suppose that $f \colon R \to Q_{ml}(R)$ is an additive map satisfying $f(x)x^m \in C$ for all $x \in R$. Then $f = 0$.*

Proof. We first assume that R is not a PI-ring. Thus, $\deg(R) = \infty$. We let S_m denote the permutation group on $\{1, 2, \ldots, m\}$. Multilinearizing $f(x)x^m \in C$, we get

$$\Big(\sum_{\sigma\in S_m} f(x_{\sigma(1)})x_{\sigma(2)}\cdots x_{\sigma(m)}\Big)x_{m+1} + \sum_{i=1}^{m} g_i(\overline{x}_{m+1}^i)x_i \in C \tag{6}$$

for all $x_i \in R$, where $g_i \colon R^m \to Q_{ml}(R)$ are m-additive maps for $i = 1, \ldots, m$. Applying [1, Corollary 2.11] to solve (6), we get

$$\sum_{\sigma\in S_m} f(x_{\sigma(1)})x_{\sigma(2)}\cdots x_{\sigma(m)} = 0$$

for all $x_i \in R$, $1 \le i \le m$. Repeating the same argument, we eventually get $f = 0$, as asserted.

We next assume that R is a PI-ring. There exists $\beta \in Z(R) \setminus \{0\}$ (see [19, Theorem 2]). Multilinearizing $f(x)x^m \in C$, we get

$$\begin{aligned}&\sum_{\sigma\in S_{m+1},\sigma(1)=1} f(x_1)x_{\sigma(2)}\cdots x_{\sigma(m+1)}\\&+\sum_{\sigma\in S_{m+1},\sigma(1)\neq 1} f(x_{\sigma(1)})x_{\sigma(2)}\cdots x_{\sigma(m+1)} \in C\end{aligned} \tag{7}$$

for all $x_i \in R$, $1 \le i \le m+1$. Replacing x_i by β for $i > 1$ in (7), we get

$$m!\Big(f(x_1)\beta^m + mf(\beta)x_1\beta^{m-1}\Big) \in C$$

for all $x_1 \in R$. Since char $(R) = 0$ or a prime $p > m$, we have

$$f(x_1)\beta + mf(\beta)x_1 \in C \tag{8}$$

for all $x_1 \in R$. Thus,

$$f(x) = \lambda x + \mu(x) \tag{9}$$

for all $x \in R$, where $\lambda = -\frac{mf(\beta)}{\beta}$ and $\mu\colon R \to C$ is an additive map. We divide the proof into two cases.

Case 1: $f\colon R \to C$. Then (8) is reduced to $f(\beta)x \in C$ for all $x \in R$. But R is not commutative. This implies $f(\beta) = 0$. Replacing x by $x+\beta$ in $f(x)x^m \in C$, we get $f(x)\Big(x^m + \sum_{i=1}^{m-1}\alpha_i x^{m-i}\beta^i\Big) \in C$ for all $x \in R$, where $\alpha_i = \binom{m}{i}$ for $i = 1, \dots, m-1$. Note that $\alpha_1\beta$ in $C \setminus \{0\}$ and $f(x)x^m \in C$. We conclude that

$$f(x)\Big(x^{m-1} + \gamma_1 x^{m-2} + \cdots + \gamma_{m-1}\Big) \in C \tag{10}$$

for all $x \in R$, where $\gamma_1, \cdots, \gamma_{m-1} \in C$. By (10) and by the fact that $(m-1)\beta$ is invertible in C, we have

$$\begin{aligned}&f(x+\beta)\Big((x+\beta)^{m-1} + \gamma_1(x+\beta)^{m-2} + \cdots + \gamma_{m-1}\Big)\\&\quad - f(x)\Big(x^{m-1} + \gamma_1 x^{m-2} + \cdots + \gamma_{m-1}\Big) \in C\,.\end{aligned}$$

and so

$$f(x)\Big(x^{m-2} + \nu_1 x^{m-3} + \cdots + \nu_{m-2}\Big) \in C$$

for all $x \in R$, where $\nu_1, \cdots, \nu_{m-2} \in C$. Repeating the same argument, we get eventually that $f(x)x \in C$ for all $x \in C$. Thus, for $x \in R$, either $f(x) = 0$ or $x \in C$. That is, R is the union of its additive subgroups $\{x \in R \mid f(x) = 0\}$ and $Z(R)$. This implies that either $f = 0$ or $R = Z(R)$. The latter case is impossible as R is not commutative. Hence, $f = 0$, as asserted.

Case 2: Consider the general case. By (9),

$$f(x)x^m = \lambda x^{m+1} + \mu(x)x^m \in C \tag{11}$$

for all $x \in R$. Let $\eta \in Z(R)$ and $x \in R$. Replacing x by ηx in (11), we get $\eta\lambda x^{m+1} + \mu(\eta x)x^m \in C$. On the other hand, $\eta\Big(\lambda x^{m+1} + \mu(x)x^m\Big) \in C$. Hence,

$$\Big(\mu(\eta x) - \eta\mu(x)\Big)x^m \in C$$

for all $x \in R$. Note that $\mu(\eta x) - \eta\mu(x) \in C$ for all $x \in R$. Applying Case 1, we get $\mu(\eta x) = \eta\mu(x)$ for all $x \in R$. This proves that $\mu\colon R \to C$ is a $Z(R)$-linear map. If $\lambda = 0$ then, by Case 1 of Lemma 2.2, $\mu = 0$ and so $f = 0$.

Suppose next that $\lambda \neq 0$. By Case 2 of Lemma 2.2, there exists an extension field F of C such that $RC \otimes_C F \cong \mathrm{M}_2(F)$ such that $\det(x)x^{m-1} \in F$ for all $x \in \mathrm{M}_2(F)$ In particular, let $x = 1 + e_{12}$. Since $m > 1$, we have

$$\det(x)x^{m-1} = 1 + (m-1)e_{12} \notin F\,,$$

as char $(R) = 0$ or a prime $p > m$. This is a contradiction. □

Proof of Theorem 2.1: We divide the proof into two cases.
Case 1: R is not a PI-ring. In this case, deg $(R) = \infty$. Multilinearizing $f(x)x^m + x^m f(x) \in C$, we get

$$\Big(\sum_{\sigma\in S_m} f\big(x_{\sigma(1)}\big)x_{\sigma(2)}\cdots x_{\sigma(m)}\Big)x_{m+1} + \sum_{i=1}^{m} g_i(\overline{x}^{\,i}_{m+1})x_i + \sum_{j=1}^{m+1} x_j h_j(\overline{x}^{\,j}_{m+1}) \in C \tag{12}$$

for all $x_1, \cdots, x_{m+1} \in R$, where $g_i, h_j\colon R^m \to Q_{ml}(R)$ are m-additive maps for $i = 1, \ldots, m$ and $j = 1, \ldots, m+1$. Apply [1, Corollary 2.11] to solve (12). Then there exist $(m-1)$-additive maps $p_j\colon R^{m-1} \to Q_{ml}(R)$, $1 \le j \le m$, and an m-additive map $\lambda\colon R^m \to C$ such that

$$\sum_{\sigma\in S_m} f\big(x_{\sigma(1)}\big)x_{\sigma(2)}\cdots x_{\sigma(m)} = \sum_{j=1}^{m} x_j p_j(\overline{x}^{\,j}_m) + \lambda(\overline{x}_m)$$

for all $x_1, \cdots, x_m \in R$. Repeating the same argument, we eventually get $f(x) = xa + \mu(x)$ for all $x \in R$, where $a \in Q_{ml}(R)$ and $\mu\colon R \to C$ is an additive map. Then

$$f(x)x^m + x^m f(x) = \Big(xax^m + x^{m+1}a\Big) + 2\mu(x)x^m \in C \tag{13}$$

for all $x \in R$. Commuting it with x, we get

$$\Big[x, xax^m + x^{m+1}a\Big] = 0 \tag{14}$$

for all $x \in R$. Note that R and $Q_{ml}(R)$ satisfy the same GPIs (see [2, Theorem 6.4.1] or [7, Theorem 2]). Thus, (14) holds for all $x \in Q_{ml}(R)$. In view of Lemma 2.1, we get $a \in C$. Then

$$\Big(ax + \mu(x)\Big)x^m \in C \tag{15}$$

for all $x \in R$. In view of Lemma 2.3, $ax + \mu(x) = 0$ for all $x \in R$. That is, $f = 0$, as asserted.

Case 2: R is a PI-ring. Multilinearizing $f(x)x^m + x^m f(x) \in C$, we get

$$\begin{aligned} &\sum_{\sigma \in S_{m+1}} f(x_{\sigma(1)})x_{\sigma(2)} \cdots x_{\sigma(m)}x_{\sigma(m+1)} \\ &+ \sum_{\sigma \in S_{m+1}} x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(m)} f(x_{\sigma(m+1)}) \in C \end{aligned} \tag{16}$$

for all $x_1, \cdots, x_{m+1} \in R$. Since R is a prime PI-ring, it follows that $Z(R) \neq 0$. Choose $\beta \in Z(R) \setminus \{0\}$. Replacing x_1 by x and x_i by β for $i > 1$, respectively, in (16), we get

$$m(m!)\beta^{m-1}\Big(f(\beta)x + xf(\beta)\Big) + 2(m!)\beta^m f(x) \in C \tag{17}$$

for all $x \in R$. Since char $(R) = 0$ or a prime $p > m > 1$, it follows from (17) that

$$f(x) = ax + xa + \mu(x) \tag{18}$$

for all $x \in R$, where $a \in Q_{ml}(R)$ and $\mu\colon R \to C$ is an additive map. Thus,

$$f(x)x^m + x^m f(x) = (ax + xa)x^m + x^m(ax + xa) + 2\mu(x)x^m \in C$$

for all $x \in R$. Commuting it with x, we get

$$\Big[x, (ax + xa)x^m + x^m(ax + xa)\Big] = 0 \tag{19}$$

for all $x \in R$. As in the proof of Lemma 2.3, (19) holds for all $RC \otimes_C F \cong \mathrm{M}_n(F)$, where $F = C$ if C is a finite field and $F = \overline{C}$, the algebraic closure of C, if C is an infinite field. Note that $n > 1$ as R is not commutative. Let $e \in \mathrm{M}_n(F)$ be an idempotent. Replacing x by e in (19), we see that $\Big[e, (ae + ea)e + e(ae + ea)\Big] = 0$. That is, $[e, a] = 0$. This proves that a commutes with any idempotents in $\mathrm{M}_n(F)$. Clearly, $a \in F$. Since $a \in RC$, the implies $a \in C$. Thus,

$$\begin{aligned} f(x)x^m + x^m f(x) &= (2ax + \mu(x))x^m + x^m(2ax + \mu(x)) \\ &= 2(2ax + \mu(x))x^m \in C \end{aligned}$$

for all $x \in R$. In view of Lemma 2.3, $2ax + \mu(x) = 0$ for all $x \in R$, i.e., $f = 0$, as asserted. □

The following characterizes additive central m-power skew commuting maps if the considered map is linear.

Theorem 2.2. *Let R be a noncommutative prime ring,* char $(R) \neq 2$. *Suppose that $f\colon R \to Q_{ml}(R)$ is a nonzero $Z(R)$-linear map satisfying $f(x)x^m + x^m f(x) \in C$ for all $x \in R$, where m is a positive integer greater than* 1. *Then* $\dim_C RC = 4$ *and* $f(x) = \lambda(x - \mathrm{Rtr}(x))$ *for all $x \in R$, where* $0 \neq \lambda \in C$.

Proof. By assumption, we have

$$[f(x), x^{2m}] = [f(x)x^m + x^m f(x), x^m] = 0$$

for all $x \in R$. Since char $(R) \neq 2$, it follows from [17, Theorem 1.1] that

$$f(x) = \lambda x + \mu(x) \tag{20}$$

for all $x \in R$, where $\lambda \in C$ and $\mu\colon R \to C$. Replacing $f(x)$ by $\lambda x+\mu(x)$ in $f(x)x^m+x^m f(x) \in C$, we have

$$(\lambda x + \mu(x))x^m \in C \tag{21}$$

for all $x \in R$ as char $(R) \neq 2$. Suppose that R is not a PI-ring. Applying the same proof given in the proof of Lemma 2.3 for the non-PI case, we get $\lambda x + \mu(x) = 0$ for all $x \in R$, i.e., $f = 0$, a contradiction. Suppose next that R is a PI-ring. Since f is $Z(R)$-linear, so is μ. By Case 2 of Lemma 2.2, $\dim_C RC = 4$ and $\mu(x) = -\lambda\,\mathrm{Rtr}(x)$ for all $x \in R$. Thus, $f(x) = \lambda(x - \mathrm{Rtr}(x))$ for all $x \in R$. □

Theorem 2.3. *Let R be a noncommutative prime ring,* char $(R) \neq 2$. *Suppose that $f\colon R \to Q_{ml}(R)$ is a nonzero $Z(R)$-linear map satisfying $f(x)x^m \in C$ for all $x \in R$, where m is a positive integer greater than* 1. *Then* $\dim_C RC = 4$ *and* $f(x) = \lambda(x - \mathrm{Rtr}\,(x))$ *for all $x \in R$, where* $0 \neq \lambda \in C$.

Proof. If R is not a PI-ring, as before by [2, Corollary 2.11] we have $f = 0$, a contradiction. Thus, R is a PI-ring. Let $0 \neq \beta \in Z(R)$. Then $f(\beta)\beta^m \in C$ and so $f(\beta) \in C$. In view of [19, Theorem 2],

$$RC = \{\frac{x}{\beta} \mid x \in R \text{ and } 0 \neq \beta \in Z(R)\}\,.$$

Since f is $Z(R)$-linear, f can be uniquely extended to a C-linear map from RC to itself. We denote such a map by the same f also. It is known that there exist finitely many $c_j, d_j \in RC$ such that

$$f(x) = \sum_j c_j x d_j \tag{22}$$

for all $x \in RC$. Clearly, $f(x)x^m \in C$ for all $x \in RC$ (see [2, Theorem 6.4.1] or [7, Theorem 2]). That is,

$$\Big(\sum_j c_j x d_j\Big)x^m \in C \tag{23}$$

for all $x \in RC$. As before, there exists an extension field F of C such that $RC \otimes_C F \cong \mathrm{M}_n(F)$, where $\dim_C RC = n^2 > 1$, f can be uniquely extended to an F-linear map from $\mathrm{M}_n(F)$ to itself and

$$f(x)x^m = \Big(\sum_j c_j x d_j\Big)x^m \in F \tag{24}$$

for all $x \in \mathrm{M}_n(F)$. Set $\lambda := -f(1) \in F$. Since $c_j, d_j \in RC$, we have $f(1) \in C$. Let e be a nontrivial idempotent in $\mathrm{M}_n(F)$. Then $f(e)e \in F$, implying $f(e)e = 0$. Note that $1 - e$ is also a nontrivial idempotent. Thus, $f(1-e)(1-e) = 0$. So

$$f(e) = -\lambda(1-e)\,. \tag{25}$$

If $\lambda = 0$, then $f = 0$ because $\mathrm{M}_n(F)$ is generated by idempotents as a vector space over F. Thus $\lambda \neq 0$. Let $x \in R$. Then $e + ex(1-e)$ and $e + (1-e)xe$ are also nontrivial idempotents. By (25) we have

$$f(e + ex(1-e)) = -\lambda\big(1 - e - ex(1-e)\big)$$
$$f(e + (1-e)xe) = -\lambda\big(1 - e - (1-e)xe\big)\,.$$

The two equalities imply that

$$f([e,x]) = \lambda[e,x] \tag{26}$$

for all $x \in \mathrm{M}_n(F)$. Let E denote the additive subgroup generated by all idempotents in $\mathrm{M}_n(F)$. Then E and $[E, \mathrm{M}_n(F)]$ are noncentral Lie ideals of $\mathrm{M}_n(F)$. In view of [11, Theorem 1.5], $[\mathrm{M}_n(F), \mathrm{M}_n(F)] \subseteq [E, \mathrm{M}_n(F)]$. Thus, by (26), we have

$$f(x) = \lambda x \tag{27}$$

for all $x \in [\mathrm{M}_n(F), \mathrm{M}_n(F)]$. By (27), $\lambda(xy - yx)^{m+1} \in F$ for all $x, y \in \mathrm{M}_n(F)$. This implies $n = 2$. Since $\mathrm{char}\,(R) \neq 2$, $\mathrm{tr}\big(2x - \mathrm{tr}(x)\big) = 0$ and so $2x - \mathrm{tr}(x) \in [\mathrm{M}_n(F), \mathrm{M}_n(F)]$. Hence,

$$f\big(2x - \mathrm{tr}(x)\big) = \lambda\big(2x - \mathrm{tr}(x)\big) \tag{28}$$

for all $x \in \mathrm{M}_2(F)$. We have

$$f\big(2x - \mathrm{tr}(x)\big) = 2f(x) - \mathrm{tr}(x)f(1) = 2f(x) + \lambda\,\mathrm{tr}(x) \tag{29}$$

for all $x \in R$. By (28) and (29), we get $f(x) = \lambda\big(x - \mathrm{tr}(x)\big)$ for all $x \in \mathrm{M}_n(F)$. In particular, $f(x) = \lambda\big(x - R\mathrm{tr}\,(x)\big)$ for all $x \in R$. □

3 Proofs of Main Theorems

Throughout, R always denotes a semiprime ring with extended centroid C. The set $\mathbf{B}$ of all idempotents of C forms a Boolean algebra with respect to the operations $e \dotplus h := e + h - 2eh$ and $e \cdot h := eh$ for all $e, h \in \mathbf{B}$. It is complete with respect to the partial order $e \leq h$ (defined by $eh = e$) in the sense that any subset S of $\mathbf{B}$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$.

A subset $\{e_\nu \in \mathbf{B} \mid \nu \in \Lambda\}$ of $\mathbf{B}$ is called orthogonal if $e_\nu e_\mu = 0$ for $\nu \neq \mu$ and is a dense subset if $\sum_{\nu\in\Lambda} e_\nu C$ is an essential ideal of C. A subset T of $Q_{ml}(R)$, where $0 \in T$,

is called orthogonally complete in the following sense: Given any dense orthogonal subset $\{e_\nu \in \mathbf{B} \mid \nu \in \Lambda\}$ of $\mathbf{B}$, there exists a one-one correspondence between T and the direct product $\prod_{\nu\in\Lambda} Te_\nu$ via the map

$$x \mapsto \langle xe_\nu \rangle \in \prod_{\nu\in\Lambda} Te_\nu \quad \text{for } x \in T .$$

Therefore, given any subset $\{a_\nu \in T \mid \nu \in \Lambda\}$, there exists a unique $a \in T$ such that $a \mapsto \langle a_\nu e_\nu \rangle$. The element a is written as $\sum_{\nu\in\Lambda}^{\perp} a_\nu e_\nu$ and is characterized by the property that $ae_\nu = a_\nu e_\nu$ for all $\nu \in \Lambda$.

In view of [2, Proposition 3.1.10], $Q_{ml}(R)$ is orthogonally complete. Moreover, P is a minimal prime ideal of $Q_{ml}(R)$ if and only if $P = \mathbf{m}Q_{ml}(R)$ for some $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$, the spectrum of $\mathbf{B}$ (i.e., the set of all maximal ideals of $\mathbf{B}$) (see [2, Theorem 3.2.15]). In particular, it follows from the semiprimeness of $Q_{ml}(R)$ that $\bigcap_{\mathbf{m}\in\operatorname{Spec}(\mathbf{B})} \mathbf{m}Q_{ml}(R) = 0$. We refer the reader to the book [2] for details.

To begin with, we need the following, which has the same proof as that of [15, Lemma 2.1].

Lemma 3.1. *Suppose that R is $m!$-torsion free. Then* $\operatorname{char}(Q_{ml}(R)/\mathbf{m}Q_{ml}(R)) = 0$ *or a prime $p > m$ for any $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$.*

Given an ideal I of R, for $q \in R$ we have $qI = 0$ if and only if $Iq = 0$. Thus, $\operatorname{Ann}_R(I) := \{q \in R \mid qI = 0\}$ is an ideal of R. Note that an ideal K of R is essential if $\operatorname{Ann}_R(K) = 0$. The following is well-known (see, for instance, [14, Lemma 2.10] with replacing Q, the Martindale symmetric ring of quotients of R, by $Q_{ml}(R)$).

Lemma 3.2. *Every annihilator ideal of $Q_{ml}(R)$ is generated by one central idempotent.*

Let $\widehat{R}$ denote the orthogonal completion of R; that is,

$$\widehat{R} = \left\{ \sum_{\alpha\in I}^{\perp} x_\alpha e_\alpha \mid \{e_\alpha \mid \alpha \in I\} \text{ is a dense orthogonal subset of } \mathbf{B} \text{ and } x_\alpha \in R\ \forall\, \alpha \in I \right\} .$$

Proof of Theorem 1.1. In view of Lemma 3.2, there exists an idempotent $e \in C$ such that $\operatorname{Ann}_{Q_{ml}(R)}\big(Q_{ml}(R)[R,R]Q_{ml}(R)\big) = eQ_{ml}(R)$. Note that $eQ_{ml}(R) \subseteq C$. Define $g\colon R \to e'Q_{ml}(R)$, where $e' := 1 - e$, by $g(x) = e'f(x)$ for all $x \in R$. Thus,

$$g(x)x^m + x^m g(x) = e'\Big(f(x)x^m + x^m f(x)\Big) \in e'C \tag{1}$$

for all $x \in R$. We claim that g can be uniquely extended to an additive map $\widehat{g}\colon \widehat{R} \to Q_{ml}(R)$, which is defined by

$$\widehat{g}\colon \sum_{\alpha\in I}^{\perp} x_\alpha e_\alpha \mapsto \sum_{\alpha\in I}^{\perp} g(x_\alpha)e_\alpha$$

for $x_\alpha \in R$. To prove the map $\widehat{g}$ to be well-defined, it suffices to prove that if $x_\alpha e_\alpha = 0$ with $x_\alpha \in R$ and $e_\alpha \in \mathbf{B}$, then $g(x_\alpha)e_\alpha = 0$. Let $y \in R$. By (1),

$$g(y + x_\alpha)(y + x_\alpha)^m + (y + x_\alpha)^m g(y + x_\alpha) \in e'C \tag{2}$$

Multiplying (2) by e_α, we see that

$$g(y+x_\alpha)y^m e_\alpha + g(y+x_\alpha)y^m e_\alpha \in e'C\,,$$

where we have used $x_\alpha e_\alpha = 0$. Since $g(y)y^m + y^m g(y) \in e'C$, we have

$$g(x_\alpha)y^m e_\alpha + g(x_\alpha)y^m e_\alpha \in e'C\,.$$

In particular, $\left[g(x_\alpha)y^m e_\alpha + g(x_\alpha)y^m e_\alpha, z\right] = 0$ for all $y, z \in R$ and so for all $y, z \in Q_{ml}(R)$ because R and $Q_{ml}(R)$ satisfy the same GPIs (see [2, Theorem 6.4.1]). In view of [13, Theorem, p.19], $\left[g(x_\alpha)ye_\alpha + g(x_\alpha)ye_\alpha, z\right] = 0$ for all $y, z \in Q_{ml}(R)$. That is,

$$g(x_\alpha)ye_\alpha + g(x_\alpha)ye_\alpha \in C \tag{3}$$

for all $y \in Q_{ml}(R)$. Commuting it with y, we get $[g(x_\alpha)e_\alpha, y^2] = 0$ for all $y \in Q_{ml}(R)$. By [13, Theorem, p.19], we conclude that $g(x_\alpha)e_\alpha \in C$. Hence, (3) is reduced to $g(x_\alpha)e_\alpha Q_{ml}(R) \subseteq C$ because R is 2-torsion free. This means that

$$g(x_\alpha)e_\alpha \in \mathrm{Ann}_{Q_{ml}(R)}\big(Q_{ml}(R)[R,R]Q_{ml}(R)\big) = eQ_{ml}(R)\,.$$

However, $g(x_\alpha)e_\alpha \in e'Q_{ml}(R)$, implying $g(x_\alpha)e_\alpha = 0$, as asserted.

Since $g(R) \subseteq e'Q_{ml}(R)$, it is clear that $\widehat{g}(\widehat{R}) \subseteq e'Q_{ml}(R)$. Moreover, we have

$$\widehat{g}(x)x^m + x^m\widehat{g}(x) \in C \tag{4}$$

for all $x \in \widehat{R}$.

Let $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$ and $y \in \widehat{R}$. We claim that $\widehat{g}(\mathbf{m}\widehat{R}) \subseteq \mathbf{m}Q_{ml}(R)$. Indeed, let $x \in \mathbf{m}\widehat{R}$. There exists $h \in \mathbf{B} \setminus \mathbf{m}$ such that $hx = 0$. By (4), we have

$$\widehat{g}(y+x)(y+x)^m + (y+x)^m\widehat{g}(y+x) \in C\,.$$

Multiplying it by h and using (4), we get

$$hy^m\widehat{g}(x) + h\widehat{g}(x)y^m \in C\,.$$

As before, we get $h\widehat{g}(x)Q_{ml}(R) \subseteq C$ and so

$$h\widehat{g}(x) \in \mathrm{Ann}_{Q_{ml}(R)}\big(Q_{ml}(R)[R,R]Q_{ml}(R)\big) = eQ_{ml}(R)\,.$$

This implies that $h\widehat{g}(x) = 0$. Thus, $\widehat{g}(x) \in \mathbf{m}Q_{ml}(R)$, as asserted. Hence, $\widehat{g}$ canonically induces an additive map $\widehat{g}_{\mathbf{m}}\colon \widehat{R}/\mathbf{m}\widehat{R} \to Q_{ml}(R)/\mathbf{m}Q_{ml}(R)$, which is defined by $\widehat{g}_{\mathbf{m}}(\overline{x}) = \overline{\widehat{g}(x)}$ for all $x \in \widehat{R}$. A direct computation shows that

$$\widehat{g}_{\mathbf{m}}(\overline{x})\overline{x}^m + \overline{x}^m\widehat{g}_{\mathbf{m}}(\overline{x}) \in \overline{C} \tag{5}$$

for all $\overline{x} \in \widehat{R}/\mathbf{m}\widehat{R}$, where $\overline{C} := C + \mathbf{m}Q_{ml}(R)/\mathbf{m}Q_{ml}(R)$. It is known that $\widehat{R}/\mathbf{m}\widehat{R}$ is a prime ring with extended centroid $\overline{C}$ and $Q_{ml}(R)/\mathbf{m}Q_{ml}(R)$ is contained in the maximal left

ring of quotients of $\widehat{R}/\mathbf{m}\widehat{R}$ (we refer the reader to the book [2] for these basic properties). It follows from Lemma 3.1 that $\operatorname{char}(\widehat{R}/\mathbf{m}\widehat{R}) = 0$ or a prime $p > m$. In view of Theorem 2.1, if $\widehat{R}/\mathbf{m}\widehat{R}$ is not commutative, then $\widehat{g}_{\mathbf{m}} = 0$; that is, $\widehat{g}(\widehat{R}) \subseteq \mathbf{m}Q_{ml}(R)$. Thus, either $[\widehat{R}, \widehat{R}] \subseteq \mathbf{m}\widehat{R}$ or $\widehat{g}(\widehat{R}) \subseteq \mathbf{m}Q_{ml}(R)$. Hence, $\widehat{g}(\widehat{R})Q_{ml}(R)[\widehat{R}, \widehat{R}]Q_{ml}(R) \subseteq \mathbf{m}Q_{ml}(R)$. But $\bigcap_{\mathbf{m}\in \operatorname{Spec}(\mathbf{B})} \mathbf{m}Q_{ml}(R) = 0$. Hence, $\widehat{g}(\widehat{R})Q_{ml}(R)[\widehat{R}, \widehat{R}]Q_{ml}(R) = 0$, implying $\widehat{g}(\widehat{R}) \subseteq eQ_{ml}(R)$ and so $\widehat{g} = 0$. In particular, $g = 0$. This implies that $f(x) = ef(x) + g(x) = ef(x) \in eQ_{ml}(R)$ for all $x \in R$. Since $eQ_{ml}(R) \subseteq C$, we conclude that $f(x)R \subseteq C$ and $Rf(x) \subseteq C$ for all $x \in R$.

Proof of Theorem 1.2. Let $f\colon R \to Q_{ml}(R)$ be an additive m-power skew commuting map. By assumption,

$$f(x)x^m + x^m f(x) = 0 \tag{6}$$

for all $x \in R$. In view of Theorem 1.1, $f(x)R \subseteq C$ and $Rf(x) \subseteq C$ for all $x \in R$. In particular, $f(x) \in C$. Thus, by (6), we get that $f(x)x^m = 0$ for all $x \in R$ as R is 2-torsion free.

Let $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$. We work in the prime ring $\overline{Q_{ml}(R)/\mathbf{m}Q_{ml}(R)}$. By Lemma 3.1, $\operatorname{char}(Q_{ml}(R)/\mathbf{m}Q_{ml}(R)) = 0$ or a prime $p > m$. Then $\overline{f(x)x^{m-1}}\,\overline{x} = \overline{0}$. Since $\overline{f(x)x^{m-1}} \in \overline{C}$, we get that either $\overline{f(x)x^{m-1}} = 0$ or $\overline{x} = \overline{0}$. In either case, $\overline{f(x)x^{m-1}} = \overline{0}$. Repeating the same argument, we get eventually that $\overline{f(x)} = \overline{0}$ for all $x \in R$. Thus, $f(R) \subseteq \mathbf{m}Q_{ml}(R)$. Since $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$ is arbitrary, we see that $f = 0$.

Applying an analogous argument given in the proof of Theorem 1.1, we have the following version of Lemma 2.3 for semiprime rings. For simplicity, we only give its statement without proof.

Theorem 3.1. *Let R be an $m!$-torsion free semiprime ring, where m is a positive integer greater than 1. Given an additive map $f\colon R \to Q_{ml}(R)$, $f(x)x^m \in C$ (or $x^m f(x) \in C$) for all $x \in R$ if and only if $f(x)R \subseteq C$.*

Finally, we turn to the proof of Theorem 1.3. The following lemma was due to Benkovič and Eremita (see [3, Lemma 2.1]) with replacing $f\colon R \to R$ by $f\colon R \to Q_{ml}(R)$. They have the same proof.

Lemma 3.3. *Let R be a prime ring and let $f\colon R \to Q_{ml}(R)$ be an additive map. If there exists a positive integer m such that $f(x)x^m = 0$ for all $x \in R$, then $f = 0$.*

For our purpose we need a generalization of Lemma 3.3 to the semiprime case.

Proposition 3.1. *Let R be a semiprime ring and let $f\colon R \to Q_{ml}(R)$ be an additive map. If there exists a positive integer m such that $f(x)x^m = 0$ for all $x \in R$, then $f = 0$.*

Proof. We first extend the additive map $f\colon R \to Q_{ml}(R)$ to an additive map $\widehat{f}\colon \widehat{R} \to Q_{ml}(R)$ by defining

$$\widehat{f}\colon \sum_{\alpha\in I}^{\perp} x_\alpha e_\alpha \mapsto \sum_{\alpha\in I}^{\perp} f(x_\alpha)e_\alpha$$

for $x_\alpha \in R$. We claim that $\widehat{f}$ is well-defined. Indeed, let $x_\alpha e_\alpha = 0$, where $x_\alpha \in R$. Then

$$f(y + x_\alpha)(y + x_\alpha)^m = 0$$

for all $y \in R$. Multiplying it by e_α and using $f(y)y^m = 0$, we see that $f(x_\alpha)e_\alpha y^m = 0$ for all $y \in R$. Since R and $Q_{ml}(R)$ satisfy the same GPIs, we get $f(x_\alpha)e_\alpha y^m = 0$ for all $y \in Q_{ml}(R)$. Replacing y by 1, we have $f(x_\alpha)e_\alpha = 0$, as asserted.

A direct computation shows that $\widehat{f}(x)x^m = 0$ for all $x \in \widehat{R}$. Let $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$. We claim that $\widehat{f}(\mathbf{m}\widehat{R}) \subseteq \mathbf{m}Q_{ml}(R)$. Indeed, let $x \in \mathbf{m}\widehat{R}$ and $y \in \widehat{R}$. There exists $h \in \mathbf{B} \setminus \mathbf{m}$ such that $hx = 0$. We have

$$\widehat{f}(y + x)(y + x)^m = 0 .$$

Multiplying it by h and using $\widehat{f}(y)y^m = 0$, we get $h\widehat{f}(x)y^m = 0$. Since y is an arbitrary element in $\widehat{R}$, we get $h\widehat{f}(x) = 0$. So $\widehat{f}(x) \in \mathbf{m}Q_{ml}(R)$, as asserted.

Thus $\widehat{f}$ induces canonically an additive map $\widehat{f}_{\mathbf{m}} \colon \widehat{R}/\mathbf{m}\widehat{R} \to Q_{ml}(R)/\mathbf{m}Q_{ml}(R)$, which is defined by

$$\widehat{f}_{\mathbf{m}}(\overline{x}) = \overline{\widehat{f}(x)}$$

for all $x \in \widehat{R}$. Clearly, $\widehat{f}_{\mathbf{m}}(\overline{x})\overline{x}^m = 0$ for all $\overline{x} \in \widehat{R}/\mathbf{m}\widehat{R}$. Since $\widehat{R}/\mathbf{m}\widehat{R}$ is a prime ring, it follows from Lemma 3.1 that $\widehat{f}_{\mathbf{m}} = 0$. That is, $\widehat{f}(\widehat{R}) \subseteq \mathbf{m}Q_{ml}(R)$. But $\bigcap_{\mathbf{m}\in\mathrm{Spec}(\mathbf{B})} \mathbf{m}Q_{ml}(R) = 0$. This implies $\widehat{f} = 0$. In particular, $f = 0$ $\square$

Proof of Theorem 1.3. Let $x \in R$. Then $[f(x)x^m + x^m f(x), x^m] = 0$ and so

$$\left[f(x), x^{2m}\right] = 0 .$$

Sine f is $Z(R)$-linear and R is a 2-torsion free semiprime ring, it follows from [12, Theorem 1.4] that $f(x) = \lambda x + \mu(x)$ for all $x \in R$, where $\lambda \in C$ and $\mu \colon R \to C$ is an additive map. Thus, $f(x)x^m + x^m f(x) = 2f(x)x^m = 0$ for all $x \in R$. So $f(x)x^m = 0$ for all $x \in R$. By Proposition 3.1, $f = 0$.

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Shriram Khanderao Nimbhorkar and Deepali Brijlal Banswal
A Note on CESS-Lattices

Abstract: Some generalizations of the concept of a CS-lattice, namely a weak CS-lattice, a CESS-lattice and a lattice satisfying the condition (P) are introduced. Various results are proved to show the relationship between these lattices and result for lattices not satisfying the condition (P) are also proved. A characterization of a lattice to be a UC-lattice is given.

Keywords: Ideal; CS-lattice (extending); weak CS-lattice (weak extending); CESS-lattice; UC-lattice; lattice satisfying the condition (P).

1 Introduction

The notion of a ring in which every complement right ideal is a direct summand was introduced by Chattres and Hajarnavis [5]. They called such a ring as a CS-ring (complements are summands). This notion was studied by several researchers in the context of modules under the names extending module or a module with C_1–property or a CS-module. These modules and their generalizations are studied by several researchers such as Harmanci and Smith [9], Dung et. al. [6], Akalan, Birkenmeir and Tercan [1], Müller and Rizvi [12], Celik, Harmanci and Smith [4] and many others.

In 1998, Celik [3] introduced a generalization of a CS-module, namely, a CESS-module. The module M is called a CESS-module if every complement in M with essential socle is a direct summand of M. He also introduced the concept of a weak CS-module and a module satisfying the condition (P). The module M is called a weak CS-module if every semisimple submodule of M is essential in a direct summand of M. The module M is said to satisfy the condition (P) if for any submodule N of M, there exists a direct summand K of M such that $Soc(K) \leq N \leq K$.

Călugăreanu [2] used lattice theory in module theory and studied several concepts from module theory in lattice theory. Keskin [10] obtained some properties of extending modules using modular lattices. Nimbhorkar and Shroff [13, 14, 15] studied respectively, ojective ideals, generalized extending ideals and Goldie extending elements in modular lattices.

In this paper, we introduce the concept of a CESS-lattice (max-semicomplement with essential socle are summands) and generalize the results of Celik [3] to a certain class of modular lattices. Smith [16, Question 1.4] posed the following question:

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Question: Whether every direct summand of a weak CS-module M is also weak CS? We formulate and answer this question in the context of certain modular lattices.

Throughout in this paper L denotes a lattice with 0.

2 Preliminaries

We recall some terms from lattice theory. These and undefined terms can be found in Grätzer [7].

Definition 2.1. *A lattice L is said to be a modular lattice if for $a, b, c \in L$ with $a \leq c$, $a \vee (b \wedge c) = (a \vee b) \wedge c$.*

Călugăreanu [2] developed the concept of an essential element in a lattice with least element 0, see also Grzeszczuk and Puczylowski [8].

Definition 2.2. *Let L be a lattice with 0. An element $a \in L$ is called an essential element if $a \wedge b \neq 0$, for any nonzero $b \in L$.*

If a is essential in $[0, b]$ then we say that a is essential in b and write $a \leq_e b$ and call b as an essential extension of a.

If $a \leq_e b$ and there is no $c \in L$ such that $a \leq_e c$ and $b < c$, then we say that b is a maximal essential extension of a.

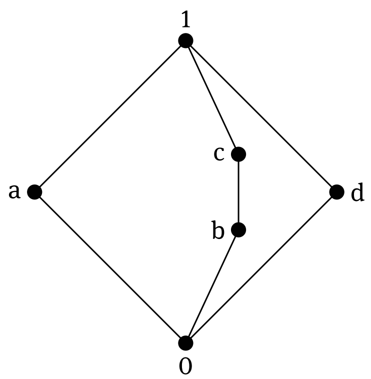

Figure 1

In the lattice L shown in Figure 1, $b \leq_e c$ and c is a maximal essential extension of b.

Definition 2.3. *Let a, b be elements of a lattice L with 0. We say that a is closed (or essentially closed) in b if a has no proper essential extension in b. If a is closed in b then we write $a \leq_{cl} b$. If a does not have a proper essential extension in L, then we say that a is closed in L.*

The concepts of a semicomplement and a maximal semicomplement are known in a lattice with 0, see Szāsz [17, p. 47]. Let $a, b \in L$, we say that a, b are semicomplements of each other if $a \wedge b = 0$.

Definition 2.4. *If $a, b \in L$ and b is a maximal element in the set $\{x \in L \mid a \wedge x = 0\}$ then we say that b is a max-semicomplement of a in L. An element $x \in L$ is called a max-semicomplement in L, if there exists a $y \in L$ such that x is a max-semicomplement of y in L.*

A max-semicomplement is called as a pseudo-complement by Călugăreanu [2, p. 39]. However in order to distinguish it from the concept of the pseudocomplement of an element in a lattice (a is called a pseudocomplement of b if b is the largest element with the property $a \wedge b = 0$) see Grätzer [7, p.63], we use the term max-semicomplement.

In the lattice shown in Figure 1, c and d are max-semicomplements of a but a does not have a pseudocomplement in L.

In the theory of modules (e.g. Lam [11, Proposition 6.24], Chatters and Hajarnavis [5, Proposition 2.2] and others), it is known that if A, B, C are modules of a ring R with $A \subseteq B \subseteq C$ and if A is closed in B and B is closed in C then A is closed in C.

The following proposition is an analog of this result, the proof of which is due to Wehrung.

Proposition 2.1. *(Nimbhorkar and Shroff [15]). Let L be a modular lattice with 0. For $a, b, c \in L$, if $a \leq_{cl} b$ and $b \leq_{cl} c$ then $a \leq_{cl} c$*

Definition 2.5. *If $a, b, c \in L$ are such that $a \vee b = c$ and $a \wedge b = 0$ then we say that a, b are direct summand of c and we write $c = a \oplus b$. We say that c is a direct sum of a and b.*

The set of all direct summands of an element $c \in L$ is denoted by $\mathfrak{D}(c)$. That is, for every $a \in \mathfrak{D}(c)$ there exists $b \in \mathfrak{D}(c)$ such that $c = a \oplus b$.

The following lemma is from Nimbhorkar and Shroff [15].

Lemma 2.1. *In a modular lattice L if $c = a \oplus b$, then a is a max-semicomplement of b in c.*

The following lemma is from Nimbhorkar and Shroff, [14, Lemma 2.1].

Lemma 2.2. *In a lattice L the following statement hold.*
(1) If $a, b, c \in L$, then $a \leq_e b$ implies $a \wedge c \leq_e b \wedge c$.
(2) If $a \leq b \leq c$, then $a \leq_e b$, $b \leq_e c$ if and only if $a \leq_e c$.

The following lemma is from Grzeszczuk and Puczylowski [8, Lemma 3].

Lemma 2.3. *Let L be a modular lattice. Suppose that $a, b, c, d \in L$ are such that $a \leq b$, $c \leq d$ and $b \wedge d = 0$. Then $a \leq_e b$, $c \leq_e d$ if and only if $a \oplus c \leq_e b \oplus d$.*

Lemma 2.4. *Let L be a modular lattice and $a, b, c \in L$, $a \leq b \leq c$. If d is a max-semicomplement of a in c then $d \wedge b$ is a max-semicomplement of a in b.*

Definition 2.6. *Let L be a lattice with 0. An element $a \in L$ is called an atom, if there does not exist any $b \in L$ such that $0 < b < a$.*

Definition 2.7. *A lattice L with the least element 0 is said to be an atomic lattice if every nonzero element of L contains an atom.*

Definition 2.8. *[2, p. 47] The join of all atoms of L, denoted by $Soc(L)$, is called the socle of the lattice L.*

For $a \in L$, $Soc(a)$ is the socle of the lattice $[0, a]$.

Definition 2.9. *Let $a, b, c \in L$ and $a \in [b, c]$; x is called a relative complement of a in $[b, c]$ if $a \wedge x = b$, $a \vee x = c$.*

A lattice L is said to be a relatively complemented lattice if every element $a \in L$ has a relative complement in L.

Throughout in this paper, wherever necessary, we assume that L satisfies one or more of the following conditions.
Condition (1): For any $a \leq b$ in L, there exists a maximal essential extension of a in b.
Condition (2): For any $a \leq b$ and for any $c \leq b$ in L with $a \wedge c = 0$, there exists a max-semicomplement $d \geq c$ of a in b.
Condition (3): If the socle is involved, $Soc(a)$ exists for any $a \in L$.
Condition (4): If $a \in L$ is a join of atoms in L then any $b \leq a$ is a join of atoms.

A familiar and important class of lattices with these properties is that of upper continuous modular lattice. In particular, the lattice of ideals of a modular lattice with 0.

The following lemma is from Nimbhorkar and shroff [15, Lemma 2.5].

Lemma 2.5. *Let L be a modular lattice satisfying the condition (2). Let $a, b \in L$ and $a \leq b$. Then a is closed in b if and only if a is a max-semicomplement of some $c \leq b$.*

The following definitions are from Călugăreanu [2].

Definition 2.10. *[2, p. 87] An element $a \in L$ is called uniform if every $b \leq a$, $b \neq 0$ is essential in $[0, a]$.*

Definition 2.11. *[2, p. 68] A modular lattice L is said to have finite Goldie dimension (Uniform dimension) n if there exists a join independent family of uniform elements $u_1, u_2, \cdots, u_n$ in L whose join is essential in L.*

The following definition is from Nimbhorkar and Shroff [15].

Definition 2.12. *Let $a, b, c \in L$ be such that $a = b \oplus c$. Then c is said to be $b-injective$ in a if for every $d \leq a$ with $d \wedge c = 0$, there exists an element $e \leq a$ such that $a = e \oplus c$ and $d \leq e$.*

In the lattice shown in Figure 1, $1 = a \oplus b$, we have $d \leq 1$ and $d \wedge b = 0$ and $d \oplus b = 1$. Hence b is $a - injective$ in 1.

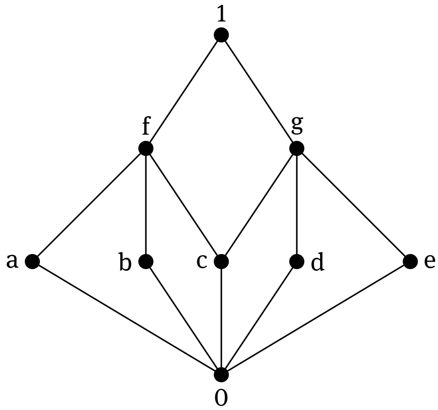

Figure 2

In the lattice L shown in Figure 2, $1 = g \oplus a$, we have $b \leq 1$ and $b \wedge a = 0$ but there is no $x \in L$ with $x \leq 1$ and $1 = x \oplus a$, $b \leq x$. Hence a is not $g - injective$ in 1.

3 CS-Lattices And Their Generalizations

The following definition is from Nimbhorkar and shroff [13].

Definition 3.1. *A bounded lattice L is called CS or extending if every nonzero element is essential in a direct summand of* 1.

A nonzero element $a \in L$ is called extending if, every nonzero $b \leq a$ is essential in a direct summand of a.

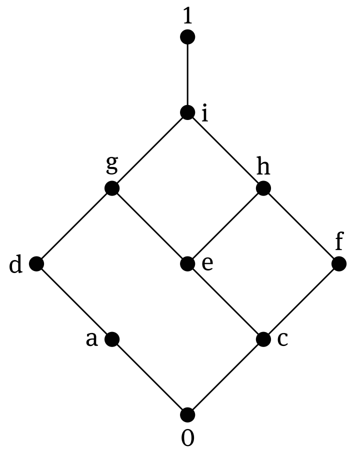

Figure 3

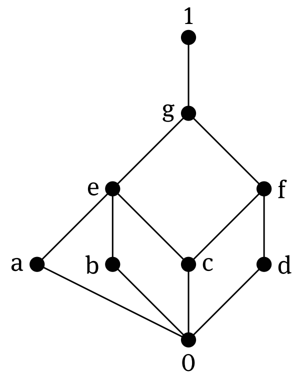

Figure 4

In the lattice L shown in Figure 3, every nonzero $x \leq g$, is essential in a direct summand of g. Hence g is extending.

In the lattice L shown in Figure 4, Consider the element g. Then $c \leq g$ but c is not essential in a direct summand of g. Hence g is not extending.

The proof of the following lemma is the same as that of Proposition 3.1 from Nimbhorkar and shroff [14].

Lemma 3.1. *Let $0 \neq a \in L$. Then the following statements are equivalent.*
(1) Every closed element $c \leq a$ in a is a direct summand of a.
(2) For every $d \leq a$, there exists a direct summand k of a such that $d \leq_e k$.

Definition 3.2. *An element $a \in L$ with 0 is called weak CS or weak extending if for every $b \leq a$ satisfying $b = Soc(b)$ is essential in a direct summand d of a.*

A bounded lattice is called weak CS, if 1 is a weak CS element.

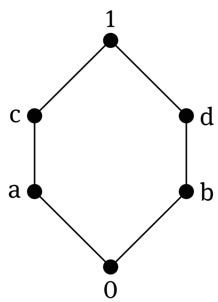

Figure 5

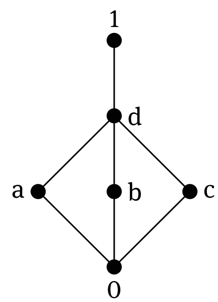

Figure 6

In the lattice L shown in Figure 5, we have $Soc(a) = a \leq_e c$, $Soc(b) = b \leq_e d$, $Soc(c) = a \leq_e c$, $Soc(d) = b \leq_e d$,. Hence L is a weak CS.

In the lattice L shown in Figure 2, $Soc(a) = a$. But a is not essential in any direct summand of 1. Hence L is not weak CS.

Celik [3] defined the concept of a CESS-module (Complements with essential socle are direct summands) which is a generalization of a CS-module. We introduce this concept in a lattice.

Definition 3.3. *An element a in a lattice L with 0 is called a CESS element, if every max-semicomlement $b \leq a$ such that $Soc(b) \leq_e b$ is a direct summand of a.*

A bounded lattice L is called a CESS-lattice if 1 is a CESS element.

In the lattice L shown in Figure 1, a, c, d are max-semicomplements in L such that $Soc(a) \leq_e a$, $Soc(c) \leq_e c$ and $Soc(d) \leq_e d$. Also, $1 = a \oplus b = a \oplus c = a \oplus d$. Hence L is a CESS lattice.

In the lattice L shown in Figure 6, a, b, c are max-semicomplements in L such that $Soc(a) \leq_e a$, $Soc(b) \leq_e b$ and $Soc(c) \leq_e c$. But none of a, b, c is a direct summand of 1. Hence L is not a CESS lattice. However, d is a CESS-element.

Definition 3.4. *Let L be a lattice with 0 and $a, b \in L$. If b is a maximal element in the set $\{x \mid x \in L$ and $a \leq_e x\}$, then we say that b is an essential closure of a in L.*

In the lattice L shown in Figure 1, c is an essential closure of b in L.

Definition 3.5. *A lattice L is called a UC-lattice if each of its nonzero element has a unique essential closure in L.*

In the lattice L given in Figure 1, every nonzero element has a unique essential closure. Hence L is a UC-lattice.

In the following lemma we prove a relationship between a CS-lattice, CESS-lattice and weak CS-lattice.

Lemma 3.2. *Let L be a lattice satisfying the conditions (1) and (3). Then every CS-lattice is a CESS-lattice and every CESS-lattice is a weak CS-lattice.*

Proof. Suppose that L is a CS-lattice. It follow that L is a CESS lattice.
Suppose that L is a CESS-lattice. Let $x = Soc(x)$, $x \neq 0$. By the condition (1), there exists a maximal essential extension y of x. Then y is closed in 1 and so y is a max-semicomplement in L. We have $x = Soc(x) \leq Soc(y) \leq y$. Now $x \leq_e y$ implies that $Soc(y) \leq_e y$. As L is CESS, y is a direct summand of 1. Thus L is weak CS. □

The following result is from Nimbhorkar and Shroff [15, Lemma 3.3].

Lemma 3.3. *Let L be a modular lattice satisfying the conditions (1) and (2). If L is a CS-lattice then any direct summand of 1 is a CS-element.*

In the next lemma, we show that a direct summand of 1 in a CESS-lattice is a CESS-element.

Lemma 3.4. *Let L be a modular lattice satisfying conditions (1), (2) and (3). If L is a CESS-lattice then any direct summand of 1 is a CESS-element.*

Proof. Let $1 = a \oplus b$. To show a is a CESS-element. Let $x \leq a$ be a max-semicomplement such that $Soc(x) \leq_e x$. Since x is a max-semicomplement, x is closed in a. Also a is closed in 1. By Proposition 2.1, x is closed in 1. As L is CESS, x is a direct summand of 1. Then $1 = x \oplus y$ for some $y \in L$. By modularity and $x \leq a$, we get $x \vee (y \wedge a) = (x \vee y) \wedge a = a$ and $x \wedge (y \wedge a) = 0$. Then $x \oplus (y \wedge a) = a$. Hence a is a CESS-element. □

The following theorem gives a necessary and sufficient condition for L to be a UC-lattice.

Theorem 3.1. *Let L be a lattice satisfying the conditions (1) and (2). The lattice L is a UC-lattice if and only if for any closed element a in L and for any $b \in L$, $a \wedge b$ is closed in b.*

Proof. Suppose that L is a UC-lattice. Let a be a closed element in L and $b \in L$. By the condition (1), $a \wedge b$ has a maximal essential extension c in b. Then $a \wedge b \leq_e c \leq b$ and so $a \wedge b = a \wedge c$. Since c is a maximal essential extension, c is closed in b and so in 1. Let d be a maximal essential extension of $a \wedge b$ in a. Then $a \wedge b \leq_e d \leq a$ and so $a \wedge b = b \wedge d$. As d is closed in a, it is closed in 1. Since L is a UC-lattice, $c = d$. Hence $a \wedge b = b \wedge d = b \wedge c = c$ Hence $a \wedge b$ is closed in b.

Conversely, suppose that the given condition holds. Let $a \in L$ and b and c be closures of a. Then $a \leq_e b$ and b is maximal with respect to this property. Similarly c is maximal with respect to $a \leq_e c$. Then b and c are closed in L. $a \leq_e b \wedge c \leq_{cl} b$ and $a \leq_e b \wedge c \leq_{cl} c$. Since $b \wedge c \leq_{cl} b$ there exists $d \leq b$ such that $d \wedge (b \wedge c) = 0$. But then $d \wedge a = 0$ contradicts $a \leq_e b$. Hence $b \wedge c = b$. Similarly $b \wedge c = c$. Thus $b = c$. □

The following result can be easily proved.

Lemma 3.5. *Let L be a lattice satisfying the conditions (3) and (4). If $a \leq b$, then $Soc(a) = a \wedge Soc(b)$.*

In the following theorem, we formulate and prove the Open problem of Smith, mentioned in the introduction.

Theorem 3.2. *Let L be a modular UC-lattice satisfying the conditions (1) to (4). If L is a weak CS-lattice then every direct summand of 1 is a weak CS-element.*

Proof. Suppose that L is a weak CS-lattice. Let $1 = a \oplus b$. To show that a is weak CS. Let $x \leq a$. We Show that $Soc(x) \leq_e y$ for some direct summand y of a. Since $x \leq 1$, $Soc(x) \leq_e d$ for some direct summand d of 1. Now $Soc(x) \leq_e t \leq a$ and t is maximal with respect to this property. Then t is closed in a and so in 1. Thus t and d are two essential closures of $Soc(x)$ in 1 and so $t = d$. Hence t is a direct summand of 1 that is

$1 = t \oplus e$. Then $a = t \vee (e \wedge a) = (t \vee e) \wedge a = 1 \wedge a$ and $t \wedge (e \wedge a) = 0$. Thus t is a direct summand of a. Hence a is a weak CS-element. □

Theorem 3.3. *Let L be a relatively complemented modular lattice satisfying the conditions (1) to (4). Suppose that any direct summand of 1 is weak CS. Let $1 = a \oplus b$, where a is b-injective. Then L is a weak CS lattice.*

Proof. Let $x \in L$ be a nonzero element satisfying $x = Soc(x)$. We show that $x \leq_e d$ for some direct summand d of 1.
Case 1: Suppose that $x \wedge a = 0$. As a is b-injective, there exists a direct summand c of 1 such that $x \leq c$ and $1 = a \oplus c$. As c is weak CS, $x \leq_e k_1$ for some direct summand k_1 of c. Let $c = k_1 \oplus k_2$. We have $1 = a \oplus c = a \oplus k_1 \oplus k_2$. Thus k_1 is a direct summand of 1 and the result holds.
Case 2: Suppose that $x \wedge a \neq 0$. As L is relatively complemented, $x = (x \wedge a) \vee t$ and $x \wedge a \wedge t = 0$ for some $t \in L$. Clearly, $t \wedge a = 0$. By the condition (4), $Soc(x \wedge a) = a \wedge x$. As a is weak CS, there exists a direct summand r_1 of a such that $x \wedge a \leq_e r_1$. Let $a = r_1 \oplus r_2$.

By the condition (4), $Soc(t) = t$. Since $t \wedge a = 0$, as in Case 1, there exists a direct summand c_1 of 1 such that $t \leq_e c_1$ and $1 = c_1 \oplus a$. As c_1 is weak CS, there exists a direct summand c_2 of c_1 such that $t \leq_e c_2$. Let $c_1 = c_3 \oplus c_4$. By Lemma 2.3, $x = (x \wedge a) \vee t \leq_e c_2 \oplus r_1$. Also, we have $1 = a \oplus c_1 = r_1 \oplus r_2 \oplus c_2 \oplus c_3$. Thus 1 is weak CS. □

We say that a bounded lattice L is a lattice with *essential socle*, if $Soc(1) \leq_e 1$. Any finite lattice is such a lattice. However, the socle of the lattice L (the descending dots indicate infinite descending chain) shown in Figure 7 is not essential in L.

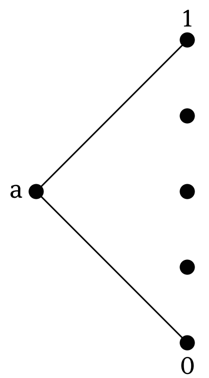

Figure 7

Theorem 3.4. *Let L be a modular, atomic and UC-lattice with essential socle and satisfying the conditions (1) to (4). Then the following statements are equivalent.*
(i) L is a weak CS-lattice.
(ii) L is a CESS-lattice.
(iii) L is a CS-lattice.

Proof. Let L be a UC lattice with essential socle that is $soc(1) \leq_e 1$.
(i) $\Rightarrow$ (ii): Suppose that L is weak CS. Let $a \in L$ be a max-semicomplement such that $Soc(a) \leq_e a$. As L is weak CS, $Soc(a) \leq_e d$ for some direct summand d of 1. Then d is closed in 1 and so by Lemma 2.5, d is a max-semicomplement.

We show that d is maximal essential extension of $Soc(a)$. Suppose that $Soc(a) \leq_e d \leq_e t$. Now $t = 1 \wedge t = (d \vee f) \wedge t = d \vee (f \wedge t)$. But $d \wedge (f \wedge t) = 0 \Rightarrow f \wedge t = 0 \Rightarrow t = d$. Thus d is a maximal essential extension of $Soc(a)$.

Since a is a max-semicomplement, there exists b such that $a \wedge b = 0$ and a is maximal with respect to this property. Let $Soc(a) \leq_e a \leq_e c$ then $b \wedge c \neq 0$ as a is max-semicomplement of b. However $a \wedge (b \wedge c) = 0$ contradicts $a \leq_e c$. Hence $a = c$. Thus a is a maximal essential extension of $Soc(a)$.

Since L is UC, we get $a = d$. Thus a is a direct summand of 1. Hence L is CESS.
(ii)$\Rightarrow$ (iii): Suppose that L is CESS.
To show that L is a CS lattice. Let $x \in L$, then by the condition (1) there exist a maximal essential extension $y \in L$ such that $x \leq_e y$. Being a maximal essential extension, y is closed in L that is y is a max-semicomplement in L. To Show: $Soc(y) \leq_e y$. Let $0 \neq a \leq y$. Since L is an atomic lattice there exist an atom $b \leq a$ and therefore $Soc(y) \wedge b = b \neq 0$. Thus $Soc(y) \leq_e y$. Since L is a CESS, y is a direct summand of 1. Hence L is a CS lattice.
(3)$\Rightarrow$ (1): By Lemma 3.2, L is a weak CS-lattice. □

Theorem 3.5. *Let $1 = \bigvee_{i=1}^{n} a_i$ be a direct sum of finitely many uniform elements a_i. Suppose that for any max-semicomplement k in L, there exists an i such that $a_i \wedge k \neq 0$. If L is UC then L is a CS lattice.*

Proof. Let k be a max-semicomplement and suppose without loss of generality, $k \wedge a_1 \neq 0$. By Theorem 3.1, $k \wedge a_1$ is closed in a_1. Since a_1 is uniform $k \wedge a_1 \leq_e a_1$ implies $k \wedge a_1 = a_1$. We have,

$$\begin{aligned} k &= k \wedge 1 = k \wedge (a_1 \vee \cdots \vee a_n) = k \wedge [(k \wedge a_1) \vee a_2 \vee \cdots \vee a_n] \\ &= (k \wedge a_1) \vee [k \wedge (a_2 \vee \cdots \vee a_n)] = a_1 \vee [k \wedge (a_2 \vee \cdots \vee a_n)] \\ &= a_1 \oplus [k \wedge (a_2 \oplus \cdots \oplus a_n)] \end{aligned}$$

By applying Theorem 3.1, $k \wedge (a_2 \oplus \cdots \oplus a_n) = m$ is closed in $a_2 \oplus \cdots \oplus a_n$. If $m \wedge a_2 \neq 0$ then $m \wedge a_2$ is closed in a_2 and a_2 is uniform implies $m \wedge a_2 = a_2$. Now,

$$\begin{aligned} k &= a_1 \oplus [k \wedge (a_2 \oplus \cdots \oplus a_n)] \\ &= a_1 \oplus [k \wedge ((m \wedge a_2) \vee \cdots \vee a_n)] \\ &= a_1 \oplus (m \wedge a_2) \oplus [k \wedge (a_3 \oplus \cdots \oplus a_n)] \\ &= a_1 \oplus a_2 \oplus [k \wedge (a_3 \oplus \cdots \oplus a_n)] \end{aligned}$$

Continuing we get a_i is a direct summand of 1. Hence L is CS. □

Lemma 3.6. *Let L be a relatively complemented lattice such that $Soc(1)$ is a maximal element of $L - \{1\}$. Then L is CESS iff L is CS.*

Proof. Suppose that L is CESS. Let k be a max-semicomplement in L such that $Soc(k) \leq_e k$. Since $Soc(1)$ is maximal in L, either $k \leq Soc(1)$ or $k \vee Soc(1) = 1$.

In the first case, since L is CESS, $k \leq d$ for some direct summand d of 1 and the result holds.

In the second case, as L is relatively complemented, $Soc(1) = (k \wedge Soc(1)) \oplus b$ for some $b \leq Soc(1)$. Then, $1 = Soc(1) \vee k = k \vee [(k \wedge Soc(1)) \vee b] = k \oplus b$. Thus L is CS.

The converse holds by Lemma 3.2 □

Definition 3.6. *A lattice L is said to satisfy the condition (P) if for $a \in L$ there exists a direct summand b of 1 such that $Soc(b) \leq a \leq b$.*

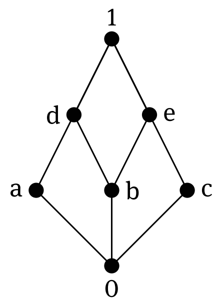

Figure 8

We note that for any nonzero nonzero x in the lattice L shown in Figure 1, there exists a direct summand y of 1 such that $Soc(y) \leq x \leq y$. For $x = a$ take $y = a$ then $Soc(a) = a \leq a$. Similarly we can show for other elements. Hence L satisfies the condition (P).

In the lattice shown in Figure 8, a is the only direct summand of 1 satisfying $a \leq d$. However $Soc(d) = a \vee b = d$ and $d \nleq a$. Thus L does not satisfy the condition (P).

Theorem 3.6. *Let L be a lattice with uniform dimension 2 such that $Soc(1)$ is a nonzero direct summand of 1. If L is not CS, then L does not satisfy the condition (P).*

Proof. By hypothesis, $1 = Soc(1) \oplus a$ for some nonzero $a \in L$. Suppose that L satisfies the condition (P). As L is not CS, there exists a max-semicomplement $k \in L$ such that k is not a direct summand of 1.

By the condition (P), there exists a direct summand b of 1 such that $Soc(b) \leq k \leq b$. Let $1 = b \oplus c$.

Case I: $b = 1$. Then $Soc(1) = Soc(b)$ and $Soc(b) \neq k$. Hence $k \wedge a \neq 0$ (otherwise, k will be a direct summand of 1). Hence $Soc(1) \oplus (k \wedge a) \leq k$. Since L has dimension 2, it follows that $k \leq_e 1$. Thus $k = 1$, a contradiction.

Case II: Suppose that $b \neq 1$. Since b is uniform, $k \leq_e b$. Since k is closed in L, it follows that $k = b$, a contradiction, which shows that L cannot satisfy condition (P). □

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Jae Keol Park and Syed Tariq Rizvi

Properties Inherited by Direct Sums of Copies of a Module

Dedicated to the memory of Professor S. M. Abul Kazim Rizvi (1918–1980)

Abstract: It is known that the quasi-retractable property of modules helps characterize a Baer module in terms of the Baer property of its endomorphism ring. We first study conditions which allow us to obtain the quasi-retractable property for certain direct sums of copies of a quasi-retractable module. Then if a ring R is right nonsingular for which the right and the left maximal rings of quotients coincide (e.g., R is semiprime PI) and M is an intermediate (R, R)-bimodule between R and the maximal right ring of quotients $Q(R)$ of R, it is shown that for any given positive integer n, $M_R^{(n)}$ is $\mathcal{K}$-cononsingular. As an application, we prove that $M_R^{(n)}$ is a Baer module if and only if $M_R^{(n)}$ is an extending module for any positive integer n. In particular, if R is a right nonsingular ring for which the right and the left maximal rings of quotients coincide and A is a right ring of quotients of R, then $A_R^{(n)}$ is a Baer module if and only if $A_R^{(n)}$ is an extending module for any positive integer n. Examples which illustrate and delimit our results are provided.

Keywords: Baer module; Baer hull; quasi-Baer module; quasi-Baer hull; extending module; extending hull; quasi-retractable; $\mathcal{K}$-cononsingular.

1 Introduction

Kaplansky introduced the notion of Baer rings in [12] which have their roots in Functional Analysis. He and many others obtained a number of interesting results on Baer rings. Recall that a ring R is called a Baer ring if the right annihilator of every nonempty subset of R is equal to eR with $e^2 = e \in R$. It can be routinely checked that a ring R is a Baer ring if and only if the left annihilator of every nonempty subset of R is equal to Rf with $f^2 = f \in R$.

More recently, the notion of a Baer ring was extended to an analogous module theoretic notion by using the endomorphism ring of a module by Rizvi and Roman [21] as follows.

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Definition 1.1. *A module M_R is called a Baer module if, for any $N_R \leq M_R$, there is $e^2 = e \in S := End(M_R)$ with $\ell_S(N) = Se$, where $\ell_S(N) = \{f \in S \mid f(N) = 0\}$. Equivalently, a module M_R is Baer if and only if for any left ideal I of S, $r_M(I) = fM$ with $f^2 = f \in S$, where $r_M(I) = \{m \in M \mid Im = 0\}$.*

We note that by Definition 1.1, a ring R is a Baer ring if and only if R_R is a Baer module.

Definition 1.2. *A module M is called extending if every submodule of M is essential in a direct summand of M*

Injective, quasi-injective, continuous, and quasi-continuous modules are extending modules. See [9] for more details for extending modules. There are close connections between an extending ring and a Baer ring (see [7]). In [21] it was shown that every $\mathcal{K}$-nonsingular extending module is Baer and that every Baer module under a dual condition becomes extending (see Theorem 3.2).

It is of a general interest to know when a certain property of a class of modules is inherited by direct sums of modules from that class. It is easy to see that while $\mathbb{Z}$ and $\mathbb{Z}_2$ are Baer $\mathbb{Z}$-modules, their direct sum is not Baer. In this paper, we study when direct sums of copies of a module with a certain property can inherit that property. It is often difficult to fully characterize when direct sums of such modules inherit the property. In most such cases, even the direct sums of copies of a module with the property may not satisfy the property. For example, if $R = \mathbb{Z}[x]$, then $(R \oplus R)_R$ is neither extending nor Baer while R_R is extending and also Baer. In this paper, our focus is on certain properties which are generally not inherited by direct sums of modules or even direct sums of copies of a module with the property.

We will find conditions which allow for direct sums of copies of a module to inherit certain properties of modules. The properties we consider to be inherited by direct sums of copies of a module are related to Baer and extending modules.

A Baer module is fully characterized in terms of its endomorphism ring and the quasi-retractable property (see Theorem 2.2). In particular, Baer modules are quasi-retractable. In Section 2, we study when the quasi-retractable property is inherited by direct sum of copies of a module. Using this result, we provide examples of quasi-retractable modules which are not Baer. Furthermore, in these examples, some modules are retractable but they have no Baer hulls.

We use the results of Section 2 in Section 3 and the close connections between Baer and extending modules (Theorem 3.2). In particular, when a ring R is right nonsingular such that the maximal right ring of quotients of R is equal to the maximal left ring of quotients of R (e.g., when R is semiprime PI, etc.) and M is an intermediate (R, R)-bimodule between R and $Q(R)$, we show that $M_R^{(n)}$ is $\mathcal{K}$-cononsingular for any positive integer n (Theorem 3.5). Using this result, in this case, we show that $M^{(n)}$ is an extending module if and only if $M^{(n)}$ is a Baer module for any positive integer n (Theorem 3.6).

Moreover, we provide an example which illustrates that Theorem 3.5 and Theorem 3.6 cannot be extended to infinite direct sums of copies of M (Example 5). As

a byproduct, if A is a right ring of quotients of R, we prove that for a given positive integer n, $A_R^{(n)}$ is a Baer module if and only if $A_R^{(n)}$ is an extending module. As an application, we compare Baer hulls and extending hulls of certain classes of modules.

All rings are assumed to have identity and all modules are assumed to be unitary. For right R-modules M_R and N_R, we use $\mathrm{Hom}(M_R, N_R)$, $\mathrm{Hom}_R(M, N)$, or $\mathrm{Hom}(M, N)$ to denote the set of all R-module homomorphisms from M_R to N_R. Let $S = \mathrm{End}(M_R)$, $\mathrm{End}_R(M)$, or $\mathrm{End}(M)$ denote the endomorphism ring of an R-module M. For an R-module homomorphism $f \in \mathrm{Hom}_R(M, N)$, we use $\mathrm{Ker}(f)$ to denote the kernel of f.

We use $E(M_R)$ or $E(M)$ to denote an injective hull of a module M_R. For a module M, $K \leq M$, $L \trianglelefteq M$, $N \leq^{\mathrm{ess}} M$, and $U \leq^{\oplus} M$ denote that K is a submodule of M, L is a fully invariant submodule of M, N is an essential submodule of M, and U is a direct summand of M, respectively. If M is an R-module, $\mathrm{Ann}_R(M)$ stands for the annihilator of M in R.

For a module M and a nonempty set Λ, let $M^{(\Lambda)}$ denote the direct sum of $|\Lambda|$ copies of M. When Λ is finite with $|\Lambda| = n$ and $n > 1$, $M^{(n)}$ stands for $M^{(\Lambda)}$. We use $\mathrm{CFM}_\Lambda(R)$ to denote the $\Lambda \times \Lambda$ column finite matrix ring over a ring R. We let $\mathrm{Mat}_n(R)$ denote the $n \times n$ matrix ring over a ring R.

For a ring R, $Q(R)$ and $Q^\ell(R)$ stand for the maximal right ring of quotients of R and the maximal left ring of quotients of R, respectively. The symbols $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{Z}_n$ ($n > 1$) denote the field of rational numbers, the ring of integers, and the ring of integers modulo n, respectively. Ideals of a ring without the adjective "left" or "right" mean two-sided ideals. Finally, for a ring R, the notation $I \trianglelefteq R$ denotes that I is an ideal of R.

2 Quasi-Retractable Modules

The notion of quasi-retractability of a module is useful for the characterization of a Baer module in terms of its endomorphism ring (see Theorem 2.2 below). In this section, for the study of Baer modules, we provide some conditions which allow for direct sums of copies of a module to become quasi-retractable. This can then be used to obtain results for direct sum of copies of a module to be Baer in view of Theorem 2.2.

We recall that a module M is said to be *retractable* if $\mathrm{Hom}_R(M, L) \neq 0$ for any $0 \neq L \leq M$.

Theorem 2.1. (i) ([23, Lemma 2.8]) *Let $\{M_i\}_{i\in\Lambda}$ be a set of retractable modules. Then $\oplus_{i\in\Lambda} M_i$ is retractable.*

(ii) ([23, Proposition 2.10]) *An arbitrary direct sum of copies of a module M is retractable if and only if M is retractable.*

Examples of retractable modules include free modules, generators, and semisimple modules. For a full characterization of a Baer module via its endomorphism ring, a more general form of retractability is defined as follows.

Definition 2.1 ([23, Definition 2.3]). *Let M_R be an R-module and $S = End_R(M)$. Then M_R is called quasi-retractable if $Hom_R(M, r_M(I)) \neq 0$ for every left ideal I of S with $r_M(I) \neq 0$ (or, equivalently, if $r_S(I) \neq 0$ for every left ideal I with $r_M(I) \neq 0$).*

If M is retractable, then M is quasi-retractable. But the converse is not true in general (see Remark 2.1). Baer modules are quasi-retractable (see Theorem 2.2).

Theorem 2.2 ([23, Theorem 2.5]). *A module M_R is Baer if and only if $End_R(M)$ is a Baer ring and M_R is quasi-retractable.*

Definition 2.2 ([21, Definition 2.5]). *A module M_R is said to be $\mathcal{K}$-nonsingular if, for all $\varphi \in End_R(M)$, $Ker(\varphi)_R \leq^{ess} M_R$ implies $\varphi = 0$.*

Every Baer module is $\mathcal{K}$-nonsingular [21, Lemma 2.15]. Also in [22], it is proved that every nonsingular module is $\mathcal{K}$-nonsingular, but not conversely in general. For example, as a $\mathbb{Z}$-module, $M = \mathbb{Z}_p$ (p is a prime integer) is $\mathcal{K}$-nonsingular, but not nonsingular. A useful type theory for $\mathcal{K}$-nonsingular extending modules has been provided in [22]. This extends the type theory for nonsingular injective modules. For more details on $\mathcal{K}$-nonsingular modules, see [21] and [22].

Lemma 2.1. *Let R be a right nonsingular ring and M an intermediate (R, R)-bimodule between R and $Q(R)$. Then $End(M_R) \cong \{q \in Q(R) \mid qM \subseteq M\}$, which is an intermediate ring between R and $Q(R)$.*

Proof. Define $\theta\colon R \to \mathrm{End}(M_R)$ such that $\theta(r)(w) = rw$ for $w \in M$ and $r \in R$. As M is an (R, R)-bimodule, $\theta(r) \in \mathrm{End}(M_R)$. For $r_1, r_2 \in R$,

$$\theta(r_1 + r_2)(w) = (r_1 + r_2)w = r_1 w + r_2 w = \theta(r_1)(w) + \theta(r_2)(w) = (\theta(r_1) + \theta(r_2))(w)$$

for $w \in M$. So $\theta(r_1 + r_2) = \theta(r_1) + \theta(r_2)$. Also $\theta(r_1 r_2)(w) = (r_1 r_2)w = \theta(r_1)\theta(r_2)(w)$ for $w \in M$. So $\theta(r_1 r_2) = \theta(r_1)\theta(r_2)$. If $\theta(r) = 0$ for $r \in R$, then $r = 0$. Therefore R is embedded as a subring of $\mathrm{End}(M_R)$ via θ.

For $f \in \mathrm{End}(M_R)$, we let $\lambda(f) \in \mathrm{End}(Q(R)_R)$ be an extension of f (note that $Q(R)_R$ is an injective hull of R_R because R is right nonsingular). We show that $S := \mathrm{End}(M_R)$ is embedded as a subring of $\mathrm{End}(Q(R)_R) = Q(R)$. For this, take $f \in \mathrm{End}(M_R)$. Let $\lambda(f) \in \mathrm{End}(Q(R)_R)$ be an extension of f. If $f_1 = f_2$ for $f_1, f_2 \in \mathrm{End}(M_R)$, then $(\lambda(f_1) - \lambda(f_2))(w) = f_1(w) - f_2(w) = 0$ for $w \in M$. Therefore $(\lambda(f_1) - \lambda(f_2))(M) = 0$, so

$$\lambda(f_1) - \lambda(f_2) \in \Delta := \{\varphi \in \mathrm{End}(Q(R)_R) \mid \mathrm{Ker}(\varphi)_R \leq^{\mathrm{ess}} Q(R)_R\}\,.$$

As $Q(R)_R$ is nonsingular, $Q(R)_R$ is $\mathcal{K}$-nonsingular by [21], and so $\Delta = 0$. Hence $\lambda(f_1) - \lambda(f_2) = 0$, i.e., $\lambda(f_1) = \lambda(f_2)$. Hence the map $\lambda\colon \mathrm{End}_R(M) \to \mathrm{End}(Q(R)_R)$ is well-defined.

Moreover, for $f, g \in \mathrm{End}(M_R)$, $\lambda(f+g) = \lambda(f)+\lambda(g)$ and $\lambda(fg) = \lambda(f)\lambda(g)$. If $\lambda(f) = 0$, then $f(M) = 0$ and hence $f = 0$. As a consequence,

$$\mathrm{End}(M_R) \cong \{q \in Q(R) \mid qM \subseteq M\} ,$$

which is a subring of $Q(R)$. Thus $\mathrm{End}(M_R)$ is embedded as a subring of $\mathrm{End}(Q(R)_R)$ via λ. So we may identify $S := \mathrm{End}(M_R)$ as a subring of $\mathrm{End}(Q(R)_R) = Q(R)$. Also note that S may be identified as an overring of R via θ. □

In the following theorem, we obtain the quasi-retractable property of direct sums of copies of a module for some classes of modules. This result will be used for investigating direct sums of Baer and extending modules.

A ring R is called a *PI-ring* if it satisfies a polynomial identity.

Theorem 2.3. (*i*) *If R is a ring and A is an intermediate ring between R and $Q(R)$, then $A_R^{(n)}$ is quasi-retractable for any positive integer n.*

(*ii*) *If R is a prime PI-ring (e.g., R is a commutative domain) and A is an intermediate ring between R and $Q(R)$, then $A_R^{(\Lambda)}$ is quasi-retractable for any nonempty set Λ. In this case, $End(A_R^{(\Lambda)}) = End(A_A^{(\Lambda)}) = CFM_\Lambda(A)$.*

(*iii*) *If R is a right nonsingular ring and M is an intermediate (R, R)-bimodule between R and $Q(R)$, then $M_R^{(n)}$ is quasi-retractable for any positive integer n.*

Proof. (i) Put $V = A_R^{(n)}$ and $S = \mathrm{End}_R(V) = \mathrm{Mat}_n(\mathrm{End}_R(A)) = \mathrm{Mat}_n(A)$. If $n = 1$, then $V = A_R$ and $S = A$. To show that A_R is quasi-retractable, let I be a left ideal of S and suppose that $r_V(I) \neq 0$. Take $0 \neq v \in r_V(I)$. Then $0 \neq v \in r_S(I)$ because $V = A = S$. Thus A_R is quasi-retractable.

Next consider when $n = 3$. Our method for the case when $n = 3$ can be applied to general case. We show that $V = A_R^{(3)}$ is quasi-retractable. For this, let I be a left ideal of $S = \mathrm{Mat}_3(A)$. Say

$$0 \neq v = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \in r_V(I) .$$

Put $0 \neq s := q_1e_{11} + q_2e_{21} + q_3e_{31} \in \mathrm{Mat}_3(A) = S$, where $e_{ij} \in S = \mathrm{Mat}_3(A)$ with 1 in the (i, j)-position and 0 elsewhere. Take $[f_{ij}] \in I$. Then $[f_{ij}]v = 0$. Hence $f_{i1}q_1 + f_{i2}q_2 + f_{i3}q_3 = 0$, for $1 \leq i \leq 3$. So $[f_{ij}]s = 0$ for all $[f_{ij}] \in I$, and thus $0 \neq s \in r_S(I)$. Hence $V = A_R^{(3)}$ is quasi-retractable.

(ii) Assume that R is a prime PI-ring. Let $F = Q(\mathrm{Cen}(R))$, where $\mathrm{Cen}(R)$ denotes the center of R. Then F is a field, and $Q(R) = RF = \mathrm{Mat}_n(D)$ for a positive integer n and a division ring D (for detail, see [20] or [25, Theorem 1.5.16, p.36 and Theorem 1.7.9, p.53]).

Now we let $V = A_R^{(\Lambda)}$ and $S = \mathrm{End}_R(V) = \mathrm{End}_R(A^{(\Lambda)})$. Then we show that $S = \mathrm{End}_A(A^{(\Lambda)}) = \mathrm{CFM}_\Lambda(A)$. For this, first note that $\mathrm{End}_A(A^{(\Lambda)}) \subseteq S$. Next, we let $f \in S$. Assume on the contrary that $f \notin \mathrm{End}_A(A^{(\Lambda)})$. Then there exist $y \in A^{(\Lambda)}$ and $q \in A$ such that $f(yq) - f(y)q \neq 0$. Since $Q(R) = RF$, we can put $q = ac^{-1}$, where $a \in R$ and

$0 \neq c \in \mathrm{Cen}(R)$. Thus

$$\begin{aligned} 0 \neq (f(yq) - f(y)q)c &= f(yq)c - f(y)a \\ &= f(yqc) - f(ya) = f(ya) - f(ya) \\ &= 0\,, \end{aligned}$$

a contradiction. So $f \in \mathrm{End}_A(A^{(\Lambda)})$. Hence $S = \mathrm{End}_A(A^{(\Lambda)}) = \mathrm{CFM}_\Lambda(A)$. By using a similar method as in part (i), we can verify that $A_R^{(\Lambda)}$ is quasi-retractable.
(iii) We prove that $V := M_R^{(n)}$ is quasi-retractable. The proof of this assertion is similar to that of part (i). We let $S = \mathrm{End}(V_R)$. Then $S = \mathrm{Mat}_n(U)$, where $U = \mathrm{End}(M_R)$. Assume I is a left ideal of S such that $r_V(I) \neq 0$.

If $n = 1$, then $V = M$ and $S = U = \mathrm{End}(M_R)$. Since $r_M(I) \neq 0$, there exists $0 \neq m \in M$ such that $Im = 0$. As $0 \neq M \subseteq Q(R)$, there is $a \in R$ with $0 \neq ma \in R$. So $I(ma) = 0$. Put $r = ma$. Then $Ir = 0$ with $0 \neq r \in R$.

We put $\Delta = \{\varphi \in \mathrm{End}(Q(R)_R) \mid \mathrm{Ker}(\varphi)_R \leq^{\mathrm{ess}} Q(R)_R\}$. Since $Q(R)_R$ is nonsingular injective, it follows that $\Delta = 0$.

Define

$$\theta\colon R \to \mathrm{End}(M_R) \text{ such that } \theta(r)(w) = rw \text{ for } w \in M \text{ and } r \in R\,.$$

If $\theta(r) = 0$ for $r \in R$, then $r = 0$ since $R \subseteq M$. It is routine to check that θ is a ring homomorphism. Thus R is embedded as a subring of $\mathrm{End}(M_R)$ via θ.

For $f \in \mathrm{End}(M_R)$, we let $\lambda(f) \in \mathrm{End}(Q(R)_R)$ be an extension of f. Then from Lemma 2.1, $\lambda\colon \mathrm{End}(M_R) \to \mathrm{End}(Q(R)_R)$ is well-defined. Now we have that

$$\lambda[I\theta(r)](R) = \lambda[I\theta(r)](1)R = [\lambda(I)\lambda(\theta(r))](1)R = (Ir)R = 0\,.$$

Therefore $\lambda[I\theta(r)] \subseteq \Delta = 0$ because $R_R \leq^{\mathrm{ess}} Q(R)_R$. So $\lambda[I\theta(r)] = 0$ and hence $I\theta(r) = 0$ since λ is one-to-one. As $0 \neq r \in R$, $0 \neq \theta(r) \in r_S(I)$.

Next let $n = 3$. Also as in part (i), Our method for this case when $n = 3$ can be applied to the general case. To show that $V = M_R^{(3)}$ is quasi-retractable, say

$$0 \neq v = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in r_V(I)\,,$$

with $m_1, m_2, m_3 \in M$. Let $[f_{ij}] \in I \subseteq S = \mathrm{Mat}_3(U)$. Since $[f_{ij}]v = 0$,

$$f_{i1}(m_1) + f_{i2}(m_2) + f_{i3}(m_3) = 0$$

for $1 \leq i \leq 3$. There exists $r \in R$ with $m_i r \in R$ and at least one of $m_i r$ nonzero, for $1 \leq i \leq 3$. Hence it follows that

$$0 = f_{i1}(m_1)r + f_{i2}(m_2)r + f_{i3}(m_3)r = f_{i1}(m_1 r) + f_{i2}(m_2 r) + f_{i3}(m_3 r)$$

for $1 \leq i \leq 3$.

Put $s = \theta(m_1 r)e_{11} + \theta(m_2 r)e_{21} + \theta(m_3 r)e_{31} \in \mathrm{Mat}_3(U) = S$, where e_{ij} is the matrix in S with 1 in the (i, j)-position and 0 elsewhere. Then $s \neq 0$. We now let

$$\alpha_i = f_{i1}\theta(m_1 r) + f_{i2}\theta(m_2 r) + f_{i3}\theta(m_3 r)$$

for $1 \leq i \leq 3$. Then

$$\begin{aligned}\lambda(\alpha_i)(R) &= [\lambda(\alpha_i)(1)]R \\ &= [f_{i1}\theta(m_1 r)(1) + f_{i2}\theta(m_2 r)(1) + f_{i3}\theta(m_3 r)(1)]R \\ &= [f_{i1}(m_1 r) + f_{i2}(m_2 r) + f_{i3}(m_3 r)]R \\ &= 0\,.\end{aligned}$$

Since $\Delta = 0$ and $\lambda(\alpha_i) \in \Delta$, we get $\lambda(\alpha_i) = 0$. As λ is one-to-one, $\alpha_i = 0$. Thus $s \neq 0$ and $[f_{ij}]s = 0$. Hence $0 \neq s \in r_S(I)$, so $r_S(I) \neq 0$. Therefore $M_R^{(3)}$ is quasi-retractable. □

Remark 2.1. *Let R be a commutative domain with the field of fractions F. For an intermediate domain A between R and F, $A_R^{(\Lambda)}$ is quasi-retractable for any nonempty set Λ by Theorem 2.3(ii). But $A_R^{(\Lambda)}$ may not be retractable. For example, $\mathbb{Q}_{\mathbb{Z}}$ is quasi-retractable. But $\mathbb{Q}_{\mathbb{Z}}$ is not retractable because Hom$(\mathbb{Q}_{\mathbb{Z}}, \mathbb{Z}_{\mathbb{Z}}) = 0$.*

Let R be a commutative noetherian domain and F be its field of fractions. Assume that $N = M_R \oplus (\oplus_{i\in\Lambda} K_i)$, where M is semisimple with a finite number of homogeneous components, and $\{K_i\}_{i\in\Lambda}$ is a set of nonzero submodules of F_R. The following theorem describes all intermediate modules between N and $E(N)$ which happen to be Baer modules.

Theorem 2.4 ([19, Theorem 2.6]). *Let R be a commutative noetherian domain, which is not a field. Assume that M is a nonzero semisimple R-module with only a finite number of homogeneous components, and $\{K_i \mid i \in \Lambda\}$ is a nonempty set of nonzero submodules of F_R, where F is the field of fractions of R. Let V_R be an essential extension of $M_R \oplus (\oplus_{i\in\Lambda} K_i)_R$. Then the following are equivalent.*
(i) V is a Baer module.
(ii) (1) $V = M \oplus W$ for some Baer essential extension W of $(\oplus_{i\in\Lambda} K_i)_R$.
(2) $Hom_R(W, M) = 0$.

As we mentioned, any Baer module is quasi-retractable (see Theorem 2.2). However, the converse does not hold in general as the following example shows. Recall that a commutative domain R is called Prüfer if every finitely generated ideal of R is projective.

Example 1. *(i) Let R be a commutative domain which is not Prüfer. Put $M = (R \oplus R)_R$. Since M is retractable, M is quasi-retractable. By* [6, Theorem 6.1.4, p.191], *$End(M_R) = Mat_2(R)$ is not Baer because R is not Prüfer. Therefore M_R is not Baer from Theorem 2.2.*
(ii) As in [19, Example 3.6], *there exists a set Λ such that $\mathbb{Z}[1/6]^{(\Lambda)}$ is not Baer as a $\mathbb{Z}$-module. But $\mathbb{Z}[1/6]^{(\Lambda)}$ is quasi-retractable as a $\mathbb{Z}$-module by Theorem 2.3(ii).*

(iii) Let $N = \mathbb{Z}_2 \oplus \mathbb{Z}[1/3]$ as a $\mathbb{Z}$-module. By modifying the proof of Theorem 2.3(i), we show that N is quasi-retractable. Let $S = End(N)$. Then

$$S = \begin{pmatrix} End(\mathbb{Z}_2) & Hom(\mathbb{Z}[1/3], \mathbb{Z}_2) \\ Hom(\mathbb{Z}_2, \mathbb{Z}[1/3]) & End(\mathbb{Z}[1/3]) \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_2 & Hom(\mathbb{Z}[1/3], \mathbb{Z}_2) \\ 0 & \mathbb{Z}[1/3] \end{pmatrix}$$

because $Hom(\mathbb{Z}_2, \mathbb{Z}[1/3]) = 0$.

Let $a \in \mathbb{Z}$, we use $\overline{a}$ to denote the image of a in $\mathbb{Z}_2$. Now for description of $Hom(\mathbb{Z}[1/3], \mathbb{Z}_2)$, we take $f \in Hom(\mathbb{Z}[1/3], \mathbb{Z}_2)$. Let $f(1) = \alpha$. Then

$$3^m f(1/3^m) = f(1) = \alpha$$

for any nonnegative integer m, hence $f(1/3^m) = \alpha$ because $\overline{3^m} = \overline{1}$. Note that

$$\mathbb{Z}[1/3] = \{n/3^m \mid n \in \mathbb{Z} \text{ and } m \text{ is a nonnegative integer}\}\,.$$

Now for $n/3^m \in \mathbb{Z}[1/3]$, $f(n/3^m) = nf(1/3^m) = n\alpha \in \mathbb{Z}_2$. Therefore the image of f is completely determined by $f(1)$. We define

$$\theta\colon Hom(\mathbb{Z}[1/3], \mathbb{Z}_2) \to \mathbb{Z}_2 \text{ by } \theta(f) = f(1) \text{ for } f \in Hom(\mathbb{Z}[1/3], \mathbb{Z}_2)\,.$$

Then θ is an additive group isomorphism. Therefore

$$S = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}[1/3] \end{pmatrix}.$$

To show that N is quasi-retractable, let I be a left ideal of S with $r_N(I) \neq 0$. Then there exists $0 \neq \beta := \begin{pmatrix} \overline{m_1} \\ m_2 \end{pmatrix} \in N$ with $\overline{m_1} \in \mathbb{Z}_2$ and $m_2 \in \mathbb{Z}[1/3]$ such that $I\beta = 0$. Thus for any $\begin{pmatrix} \overline{x_{11}} & \overline{x_{12}} \\ 0 & x_{22} \end{pmatrix} \in I$, we have that

$$\begin{pmatrix} \overline{x_{11}} & \overline{x_{12}} \\ 0 & x_{22} \end{pmatrix} \begin{pmatrix} \overline{m_1} \\ m_2 \end{pmatrix} = \begin{pmatrix} \overline{x_{11}}\,\overline{m_1} + \overline{x_{12}}\,\overline{m_2} \\ x_{22}m_2 \end{pmatrix} = 0\,.$$

Let

$$s = \begin{pmatrix} \overline{0} & \overline{m_1} \\ 0 & m_2 \end{pmatrix} \in S\,.$$

Then $s \neq 0$, since $\beta \neq 0$. Now for any $\begin{pmatrix} \overline{x_{11}} & \overline{x_{12}} \\ 0 & x_{22} \end{pmatrix} \in I$,

$$\begin{pmatrix} \overline{x_{11}} & \overline{x_{12}} \\ 0 & x_{22} \end{pmatrix} s = \begin{pmatrix} \overline{x_{11}} & \overline{x_{12}} \\ 0 & x_{22} \end{pmatrix} \begin{pmatrix} \overline{0} & \overline{m_1} \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} \overline{0} & \overline{x_{11}}\,\overline{m_1} + \overline{x_{12}}\,\overline{m_2} \\ 0 & x_{22}m_2 \end{pmatrix} = 0$$

by the preceding argument. So $0 \neq s \in r_S(I)$, and hence $r_S(I) \neq 0$. Therefore N is quasi-retractable.

Because $Hom(\mathbb{Z}[1/3], \mathbb{Z}_2) \neq 0$, Theorem 2.4 yields that $\mathbb{Z}_2 \oplus \mathbb{Z}[1/3]$ is not Baer. as a $\mathbb{Z}$-module.

3 $\mathcal{K}$-Cononsingular Modules and Applications

When a ring R is right nonsingular such that $Q(R) = Q^{\ell}(R)$ (e.g., semiprime PI, semiprime almost PI, or semiprime intrinsically PI) and M is an intermediate (R, R)-bimodule between R and $Q(R)$, we show that for any given positive integer n, $M_R^{(n)}$ is $\mathcal{K}$-cononsingular. Using this result, we prove that $M_R^{(n)}$ is Baer if and only if $M_R^{(n)}$ is extending.

For Baer module properties, and the relationship between Baer modules and extending modules, we start with the following.

Theorem 3.1 ([21, Theorem 2.17]). *Let M be a Baer module. Then every direct summand of M is also a Baer module.*

Definition 3.1 ([21, Definition 2.7]). *A module M_R is called $\mathcal{K}$-cononsingular if, for $N_R \leq M_R$, $\ell_S(N) = 0$ implies $N_R \leq^{ess} M_R$, where $S = End_R(M)$.*

It is easy to see that every extending module is $\mathcal{K}$-cononsingular. There are close connections between an extending ring and a Baer. Indeed, it is proved in [7] that a ring R is right extending and right nonsingular if and only if R is Baer ring and right cononsingular. The following result shows an analogous strong bond between extending modules and Baer modules.

Theorem 3.2 ([21, Theorem 2.12]). *A module M_R is extending and $\mathcal{K}$-nonsingular if and only if M_R is Baer and $\mathcal{K}$-cononsingular.*

Extending the notion of a Baer ring, a quasi-Baer ring was introduced by Clark in [8]. A ring for which the left annihilator of every ideal is generated by an idempotent as a left ideal is called a *quasi-Baer ring*. It was initially defined by Clark to help characterize a finite dimensional algebra over an algebraically closed field F to be a twisted semigroup algebra of a matrix units semigroup over F. It was also shown in [8] that any finite distributive lattice is isomorphic to a certain sublattice of the lattice of all ideals of an artinian quasi-Baer ring. It is routine to show that a ring R is quasi-Baer if and only if the right annihilator of every ideal of R is generated, as a right ideal, by an idempotent.

Clearly every Baer ring is a quasi-Baer ring while the converse is not true in general, for example, the 2×2 upper triangular matrix ring over $\mathbb{Z}$ is a quasi-Baer ring, but it is not a Baer ring. It is also obvious that two notions coincide for a commutative ring and for a reduced ring. See [6] for more details on quasi-Baer rings.

Recently, the notion of a quasi-Baer ring was extended to an analogous module theoretic notion via the endomorphism ring of a module by Rizvi and Roman [21] in view of the Morita context as follows.

Definition 3.2 ([21, Definition 3.2]). *A right R-module M is said to be a quasi-Baer module if for each $N \trianglelefteq M$, $\ell_S(N) = Se$ for some $e^2 = e \in S := End_R(M)$, where $\ell_S(N) = \{f \in$*

$S \mid f(N) = 0\}$. Equivalently, M is a quasi-Baer module if and only if, for any ideal J of S, $r_M(J) = fM$ for some $f^2 = f \in S$, where $r_M(J) = \{m \in M \mid Jm = 0\}$.

We note that a ring R is quasi-Baer if and only if R_R is a quasi-Baer module. Recall from [4] that a module M is said to be *FI-extending* if every fully invariant submodule of M is essential in a direct summand of M. Obviously, every extending module is FI-extending. See [4] and [6] for more details on FI-extending modules. It is shown in [21] that there are close connections between a quasi-Baer module and an FI-extending module.

Theorem 3.3. (i) ([21, Theorem 4.1]) *Let M_R be a (quasi-)Baer module. Then $\mathrm{End}_R(M)$ is a (quasi-)Baer ring.*

(ii) ([14, Proposition 2.11]) *Let M_R be a quasi-retractable and $\mathrm{End}_R(M)$ be a quasi-Baer ring. Then M_R is a quasi-Baer module.*

(iii) ([21, Theorem 3.17]) *Any direct summand of a (quasi-)Baer module is a (quasi-) Baer module.*

(iv) ([21, Corollary 3.20]) *A free module over a quasi-Baer ring is a quasi-Baer module.*

Example 2. (i) *Every Baer module is a quasi-Baer module.*

(ii) *All semisimple modules are (quasi-)Baer modules.*

(iii) *Every nonsingular injective (even extending) module is a Baer module (see Theorem 3.2).*

(iv) *Any free R-module $R_R^{(\Lambda)}$ with Λ a nonempty set, over a quasi-Baer ring R which is not a Baer ring (e.g., $R = T_2(\mathbb{Z})$, the 2×2 matrix ring over $\mathbb{Z}$), is a quasi-Baer module but it is not a Baer module.*

Let $\mathcal{B}(Q(R))$ be the set of all central idempotents of $Q(R)$. We recall from [3] that the ring $R\mathcal{B}(Q(R))$, the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$, is called the *idempotent closure* of a ring R. The idempotent closure is critically used for a decomposition into a direct product of indecomposable rings or into a direct product of prime rings. Also the idempotent closure is a key ingredient to show that a boundedly centrally closed unital C^*-algebra is precisely a unital C^*-algebra, which is a quasi-Baer ring (see [6, Section 3, Chapter 10]).

Lemma 3.1 ([5, Theorem 3.3]). *Let R be a semiprime ring and A be an intermediate ring between R and $Q(R)$. Then A is a quasi-Baer ring if and only if $R\mathcal{B}(Q(R)) \subseteq A$.*

Theorem 3.4. *Let R be a semiprime ring and A be an intermediate ring between R and $Q(R)$. Then the following are equivalent.*

(i) *$A_R^{(n)}$ is a quasi-Baer module for each positive integer n.*

(ii) *$\mathrm{Mat}_n(A)$ is a quasi-Baer ring for each positive integer n.*

(iii) *A_R is a quasi-Baer module.*

(iv) *A is a quasi-Baer ring.*

(v) *$R\mathcal{B}(Q(R)) \subseteq A$.*

Thereby, for each given positive integer n, $R\mathcal{B}(Q(R))$ is the smallest intermediate ring between R and $Q(R)$ such that $R\mathcal{B}(Q(R))_R^{(n)}$ is a quasi-Baer module.

Proof. (i)⇔(ii) By Theorem 2.3(i), $A_R^{(n)}$ is quasi-retractable for any positive integer n. So $A_R^{(n)}$ is a quasi-Baer module if and only if $\mathrm{End}_R(A^{(n)}) = \mathrm{Mat}_n(A)$ is a quasi-Baer ring from Theorem 3.3(ii).
(iii)⇔(iv) follows similarly to the proof (i)⇔(ii).
(ii)⇔(iv) follows from [6, Theorem 3.2.12, p.76].
(iv)⇔(v) follows from Lemma 3.1. □

The following example shows that the Baer module version of Theorem 3.4 does not hold true. For a module M_R, let $Z(M_R)$ denote the singular submodule of M_R.

Example 3. *By* [17], *there exists a prime ring R which is not right nonsingular. Let A be any intermediate ring between $R\mathcal{B}(Q(R)) = R$ and $Q(R)$. Then A_R is not right nonsingular. Thus there exists $0 \neq y \in Z(A_R)$. So $yK = 0$ for some $K_R \leq^{ess} R_R$. Since $R_R \leq^{ess} A_R$, we have that $K_R \leq^{ess} A_R$. Hence $KA_R \leq^{ess} A_R$, and so $KA_A \leq^{ess} A_A$. Now $yKA = 0$ as $yK = 0$. Thus $0 \neq y \in Z(A_A)$. So the ring A is not right nonsingular. Therefore $\mathrm{End}_R(A) = A$ is not a Baer ring because a Baer ring is right (left) nonsingular. From Theorem 2.2, A_R is not a Baer module because A_R is quasi-retractable from Theorem 2.3(i). Thus for each positive integer n, $A_R^{(n)}$ is not a Baer module by Theorem 3.1.*

According to [2]: (i) a ring R is called an *almost PI-ring* if every prime factor ring of R is a PI-ring; and (ii) a ring R is called an *intrinsically PI-ring* if every nonzero ideal contains a nonzero PI-ideal of R. For a semprime ring the following implications subsist and are not reversible (for more detail, see [2]):

commutative ⇒ PI ⇒ almost PI ⇒ intrinsically PI.

When R is a semiprime PI-ring, from [11] R is right (and left) nonsingular, and by [16] and [24] $Q(R) = Q^{\ell}(R)$. Furthermore, when R is semiprime almost PI, or semiprime intrinsically PI, it is shown in [2, Theorem 3.10] that R is right (and left) nonsingular and $Q(R) = Q^{\ell}(R) = Q^{s}(R)$, where $Q^{s}(R)$ denotes the symmetric ring of quotients of R.

For a ring R, R_R is $\mathcal{K}$-cononsingular if and only if R is right cononsingular. Also for a commutative semiprime ring R, $R_R^{(n)}$ is $\mathcal{K}$-cononsingular for every positive integer n. From [21, Lemma 2.13], every extending module is $\mathcal{K}$-cononsingular. But the converse is not true. Indeed, let $R = \mathbb{Z}[x]$. Then $(R \oplus R)_R$ is $\mathcal{K}$-cononsingular since the ring R is commutative semiprime. But $(R \oplus R)_R$ is not extending.

In spite of preceding examples, not much is known about $\mathcal{K}$-cononsingular modules except [10]. The following example shows that the direct sum of two $\mathcal{K}$-cononsingular modules need not be $\mathcal{K}$-cononsingular.

Example 4 ([10, Example 2.8]). *Let p be a prime integer. Then $\mathbb{Z}_p$ and $\mathbb{Q}$ are $\mathcal{K}$-cononsingular as $\mathbb{Z}$-modules. But $\mathbb{Z}_p \oplus \mathbb{Q}$ is not $\mathcal{K}$-cononsingular.*

However, in the following, we obtain a direct sum of a finite copies of a certain module is $\mathcal{K}$-cononsingular module. Our next result about $\mathcal{K}$-cononsingularity of $M_R^{(n)}$ is motivated by Example 4. This will be useful to obtain conditions for $M_R^{(n)}$ to be Baer (extending) in Theorem 3.6.

Theorem 3.5. *Let R be a right nonsingular ring such that $Q(R) = Q^{\ell}(R)$ (e.g., R is semiprime intrinsically PI). If M is an intermediate (R,R)-bimodule between R and $Q(R)$, then $M_R^{(n)}$ is $\mathcal{K}$-cononsingular for any positive integer n.*

Proof. We put $U = \mathrm{End}(M_R)$. Then $S := \mathrm{End}_R(M^{(n)}) = \mathrm{Mat}_n(\mathrm{End}(M_R)) = \mathrm{Mat}_n(U)$. From Lemma 2.1,

$$U \cong T := \{q \in Q(R) \mid qM \subseteq M\}\ ,$$

an intermediate ring between R and $Q(R)$. Since $Q(R) = Q^{\ell}(R)$, $Q(T) = Q(R)$ and $Q^{\ell}(T) = Q(R)$ hold. Therefore $Q(T) = Q^{\ell}(T)$, so $Q(U) = Q^{\ell}(U)$. To show that $M_R^{(n)}$ is $\mathcal{K}$-cononsingular, assume that $N_R \leq M_R^{(n)}$ such that $\ell_S(N) = 0$. So $\ell_{\mathrm{Mat}_n(U)}(N) = 0$. Now define

$$\theta\colon R \to \mathrm{End}(M_R) \ \text{ such that } \ \theta(r)(m) = rm \ \text{ for } \ r \in R \ \text{ and } \ m \in M\ .$$

Then from Lemma 2.1, θ is a ring monomorphism.

Consider the case when $n = 1$. Then $N_R \leq M_R$. We show that $\ell_{Q(R)}(N) = 0$. If not, there exists $0 \neq q \in Q(R)$ such that $qN = 0$. Thus for any $n \in N$ such that $qn = 0$. Since $Q(R) = Q^{\ell}(R)$, there is $r \in R$ such that $0 \neq rq \in R$. Now $\theta(rq)(n) = (rq)n = r(qn) = 0$. Since θ is one-to-one, $0 \neq \theta(rq) \in U$. Hence $0 \neq \theta(rq) \in \ell_U(N)$, which is a contradiction. Hence $\ell_{Q(R)}(N) = 0$.

Next, consider the case when $n = 3$. Then $N_R \leq M_R^{(3)}$. We show that $\ell_{\mathrm{Mat}_3(Q(R))}(N) = 0$. For this, assume on the contrary that $\ell_{\mathrm{Mat}_3(Q(R))}(N) \neq 0$. Take $0 \neq [q_{ij}] \in \ell_{\mathrm{Mat}_3(Q(R))}(N)$. Then for any

$$n = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in N \ \text{ with } m_1, m_2, m_3 \in M\ ,$$

we have

$$0 = [q_{ij}]n = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

and so $q_{i1}m_1 + q_{i2}m_2 + q_{i3}m_3 = 0$ for $1 \leq i \leq 3$. Since $[q_{ij}] \neq 0$, we may assume that the first row of the matrix $[q_{ij}]$ is nonzero. So at least one of q_{11}, q_{12}, and q_{13} is nonzero. Since $Q(R) = Q^{\ell}(R)$, there exists $r \in R$ such that $rq_{ij} \in R$ for $1 \leq i, j \leq 3$, and at least one of rq_{11}, rq_{12}, and rq_{13} is nonzero. Also

$$rq_{i1}m_1 + rq_{i2}m_2 + rq_{i3}m_3 = 0 \text{ for } 1 \leq i \leq 3\ .$$

As each $rq_{ij} \in R$, $\theta(rq_{ij})(m_j) = rq_{ij}m_j$. Thus it follows that

$$\theta(rq_{i1})(m_1) + \theta(rq_{i2})(m_2) + \theta(rq_{i3})(m_3) = rq_{i1}m_1 + rq_{i2}m_2 + rq_{i3}m_3 = 0$$

for $1 \leq i \leq 3$. So $[\theta(rq_{ij})]n = 0$ for all $n \in N$. Note that $[\theta(rq_{ij})] \in \mathrm{Mat}_3(U)$. Further, $[\theta(rq_{ij})] \neq 0$ because θ is one-to-one and at least one of rq_{11}, rq_{12}, and rq_{13} is nonzero. Therefore $0 \neq [\theta(rq_{ij})] \in \ell_{\mathrm{Mat}_3(U)}(N)$, which is a contradiction. So $\ell_{\mathrm{Mat}_3(Q(R))}(N) = 0$.

Our preceding method can be applied for the case when n is any positive integer. Hence $\ell_{\mathrm{Mat}_n(Q(R))}(N) = 0$ for any positive integer n. As $Q(R)_R^{(n)}$ is injective and nonsingular, $Q(R)_R^{(n)}$ is $\mathcal{K}$-nonsingular by [21] and so $Q(R)_R^{(n)}$ is Baer and $\mathcal{K}$-cononsingular from Theorem 3.2. Further, since $\mathrm{End}(Q(R)_R^{(n)}) = \mathrm{Mat}_n(Q(R))$ and $\ell_{\mathrm{Mat}_n(Q(R))}(N) = 0$, it follows that $N_R \leq^{\mathrm{ess}} Q(R)_R^{(n)}$. Hence $N_R \leq^{\mathrm{ess}} M_R^{(n)}$. As a consequence, we have that $M_R^{(n)}$ is $\mathcal{K}$-cononsingular. □

In the following, we provide a class of modules N such that N is Baer if and only if N is extending, as a consequence of Theorem 3.5.

Theorem 3.6. *Assume that R is a right nonsingular ring such that $Q(R) = Q^\ell(R)$ (e.g., R is semiprime intrinsically PI). Let M be an intermediate (R, R)-bimodule between R and $Q(R)$, and n be a given positive integer. Then the following are equivalent.*
(i) $M_R^{(n)}$ is a Baer module.
(ii) $M_R^{(n)}$ is an extending module.
(iii) $\mathrm{Mat}_n(\mathrm{End}(M_R))$ is a Baer ring.
(iv) $\mathrm{Mat}_n(\mathrm{End}(M_R))$ is a right extending ring.
(v) $\mathrm{Mat}_n(\mathrm{End}(M_R))$ is a left extending ring.

Proof. (i)⇔(ii) From Theorem 3.5, $M_R^{(n)}$ is $\mathcal{K}$-cononsingular. Since $M_R^{(n)}$ is nonsingular, $M_R^{(n)}$ is $\mathcal{K}$-nonsingular by [21]. Thus (i)⇔(ii) follows from Theorem 3.2.
(i)⇔(iii) By Theorem 2.3(iii), $M_R^{(n)}$ is quasi-retractable. So $M_R^{(n)}$ is Baer if and only if $\mathrm{End}(M_R^{(n)}) = \mathrm{Mat}_n(\mathrm{End}(M_R))$ is a Baer ring from Theorem 2.2.
(iii)⇔(iv)⇔(v) Note that $S := \mathrm{End}(M_R^{(n)}) = \mathrm{Mat}_n(\mathrm{End}(M_R)) = \mathrm{Mat}_n(U)$, where $U = \mathrm{End}(M_R)$. From Lemma 2.1,

$$U \cong T := \{q \in Q(R) \mid qM \subseteq M\} .$$

Further, T is an intermediate ring between R and $Q(R)$. Thus $Q(T) = Q(R)$ and $Q^\ell(T) = Q^\ell(R)$. As $Q(R) = Q^\ell(R)$, $Q(T) = Q^\ell(T)$. Hence $Q(U) = Q^\ell(U)$ because $U \cong T$. From [27] and the fact that $Q(U) = Q^\ell(U)$, it follows that

$$Q(S) = Q(\mathrm{Mat}_n(U)) = \mathrm{Mat}_n(Q(U)) = \mathrm{Mat}_n(Q^\ell(U)) = Q^\ell(\mathrm{Mat}_n(U)) = Q^\ell(S) .$$

Since R is right nonsingular, $Q(R)$ is (von Neumann) regular, and so $Q(T)$ is regular. As $U \cong T$, $Q(U) \cong Q(T)$ so $Q(U)$ is regular. Therefore $Q(S) = \mathrm{Mat}_n(Q(U))$ is regular. Hence $S = \mathrm{Mat}_n(\mathrm{End}(M_R))$ is right nonsingular. Therefore [6, Corollary 3.3.3, p.90] yields that the ring S is Baer if and only if S is right extending if and only if S is left extending. □

The next example shows that Theorem 3.5 and Theorem 3.6 cannot be extended to the case of a direct sum of infinite copies of an intermediate (R, R)-bimodule M between R and $Q(R)$, where R is a right nonsingular ring such that $Q(R) = Q^\ell(R)$.

Example 5. *Let $R = \mathbb{Z}$ and let Λ be a countably infinite set. Then by* [6, Example 4.5.2(v), p.125], *$R_R^{(\Lambda)}$ is a Baer module. Because $R_R^{(\Lambda)}$ is not extending (see* [9, p.56]*), $R_R^{(\Lambda)}$ is not $\mathcal{K}$-cononsingular by Theorem 3.2.*

Assume that R is a commutative domain with the field of fractions F, and M_R is an intermediate module between R_R and F_R. Then notice that $\mathrm{End}(M_R)$ is isomorphic to an intermediate domain between R and F from Lemma 2.1.

Corollary 3.1. *Assume that R is a commutative domain with the field of fractions F. Let $R_R \leq M_R \leq F_R$. Then the following are equivalent.*
(i) $M_R^{(n)}$ is a Baer module for every positive integer n.
(ii) $M_R^{(n)}$ is an extending module for every positive integer n.
(iii) $End_R(M)$ is a Prüfer domain.
In particular if R is a Prüfer domain, then $M_R^{(n)}$ is a Baer and extending module for every positive integer n.

Proof. The equivalence of (i), (ii), and (iii) follows from [6, Theorem 6.1.4, p.191] and Theorem 3.6. In particular if R is a Prüfer domain, then $\mathrm{End}(M_R)$ is also a Prüfer domain since $\mathrm{End}(M_R)$ is isomorphic to an intermediate domain between R and F by Lemma 2.1. So for each positive integer n, $\mathrm{End}(M_R^{(n)}) = \mathrm{Mat}_n(\mathrm{End}(M_R))$ is a Baer and right (and left) extending ring by [6, Theorem 6.1.4, p.191]. From Theorem 3.6, $M_R^{(n)}$ is Baer and extending for every positive integer n. □

Corollary 3.2. *Let R be a right nonsingular ring such that $Q(R) = Q^{\ell}(R)$. Assume that A is an intermediate ring between R and $Q(R)$, and let n be a given positive integer. Then the following are equivalent.*
(i) $A_R^{(n)}$ is a Baer module.
(ii) $A_R^{(n)}$ is an extending module.
(iii) $Mat_n(A)$ is a Baer ring.
(iv) $Mat_n(A)$ is a right extending ring.
(v) $Mat_n(A)$ is a left extending ring.

Proof. We note that $\mathrm{End}(A_R^{(n)}) = \mathrm{Mat}_n(\mathrm{End}(A_R)) = \mathrm{Mat}_n(A)$. Thus the equivalence of (i)–(v) follows immediately from Theorem 3.6. □

Definition 3.3 ([6, Definition 8.4.1, p.310]). *Let M_R be a module. We fix an injective hull $E(M_R)$ of M_R. Let $\mathfrak{M}$ be a class of modules. We call, when it exists, a module H_R the $\mathfrak{M}$ hull of M_R if H_R is the smallest extension of M_R in $E(M_R)$ that belongs to $\mathfrak{M}$.*

Let M be a module. By Definition 3.3, the *Baer* (resp., *extending,* etc.) *hull* of M is the smallest Baer (resp., extending, etc.) essential extension of M in $E(M)$.

Corollary 3.3. *Let R be a right nonsingular ring such that $Q(R) = Q^{\ell}(R)$. Assume that M is an intermediate (R, R)-bimodule between R and $Q(R)$, and n is a given positive integer. If M_R is Baer, then the following are equivalent.*
(i) $M_R^{(n)}$ has a Baer hull.

(ii) $M_R^{(n)}$ *has an extending hull.*
In this case, $M_R^{(n)}$ *itself is both the Baer hull and the extending hull of* $M_R^{(n)}$*.*

Proof. (i)⇒(ii) First, suppose $n = 1$. By assumption, M_R is Baer. From Theorem 3.6, M_R is extending. So M_R itself is the extending hull of M_R.

Next assume $n \geq 2$. Let $\Lambda = \{1, 2, \ldots, n\}$. By hypothesis, $M_R^{(n)}$ has a Baer hull, say U_R. Note that $E(M_R) = Q(R)$. In this case,

$$(M \oplus Q(R)^{(\Lambda\setminus\{i\})})_R \ , \text{ where } 1 \leq i \leq n \ ,$$

is a Baer module from [6, Theorem 4.2.18, p.107] (see also [15, Theorem 2.16]). Therefore $U \subseteq \cap_{i\in\Lambda}(M \oplus Q(R)^{\Lambda\setminus\{i\}})$. Hence $U = M^{(n)}$ and thus $M_R^{(n)}$ is Baer. So $M_R^{(n)}$ is extending by Theorem 3.6. Therefore $M_R^{(n)}$ itself is the extending hull of $M_R^{(n)}$.

(ii)⇒(i) Suppose first that $n = 1$. By hypothesis, M_R is Baer, so M_R itself is the Baer hull of M_R. Let $n \geq 2$. Put $\Lambda = \{1, 2, \ldots, n\}$ as above. Assume that $M_R^{(n)}$ has an extending hull, say V_R.

By [1, Proposition 1.8(ii) and Corollary 3.3(i)], each $(M \oplus Q(R)^{(\Lambda\setminus\{i\})})_R$, $1 \leq i \leq n$, is extending. So

$$V \subseteq \cap_{i\in\Lambda}(M \oplus Q(R)^{(\Lambda\setminus\{i\})}) = M^{(n)} \ .$$

Hence $V = M^{(n)}$ and thus $M_R^{(n)}$ is extending, thus $M_R^{(n)}$ is Baer from Theorem 3.6. Therefore $M_R^{(n)}$ itself is the Baer hull of $M_R^{(n)}$. □

Let R be a commutative domain with the field of fractions F. A submodule K of F_R is called a *fractional ideal* of R if $rK \subseteq R$ for some $0 \neq r \in R$. Thus $K_R \cong (rK)_R$ and rK is an ideal of R.

For a fractional ideal K of R, we put $K^{-1} = \{q \in F \mid qK \subseteq R\}$, which is called the *inverse* of K. We say that a fractional ideal K is *invertible* if $KK^{-1} = R$. It is well-known that for a nonzero ideal I of a commutative domain R, I_R is projective if and only if $II^{-1} = R$. In this case, I_R is finitely generated. Thus for each nonzero ideal I of a Dedekind domain R, it follows that $II^{-1} = R$ because I_R is projective.

Furthermore, every nonzero fractional ideal of a Dedekind domain is invertible. We note that a Dedekind domain is noetherian because every ideal is projective (hence every ideal is finitely generated). See [26, Chapter 6] for more details on Dedekind domains.

If I is an invertible ideal of a commutative domain R, then we let

$$I^{-2} = I^{-1}I^{-1} \ , \quad I^{-3} = I^{-1}I^{-1}I^{-1} \ , \quad \text{and so on.}$$

For convenience, we put $I^0 = R$.

Assume that R is a Dedekind domain. Then for nonzero ideals $I_1, I_2, \ldots, I_n$ of R, it can be checked that $(I_1 I_2 \cdots I_n)^{-1} = I_n^{-1} \cdots I_2^{-1} I_1^{-1}$ (see [19, Lemma 2.8]).

We denote the Baer hull of a module M by $\mathbf{B}(M)$ when it exists.

Theorem 3.7 ([19, Theorem 2.13]). *Let R be a Dedekind domain. Assume that M is an R-module with $Ann_R(M) \neq 0$, and $\{K_1, K_2, \dots, K_m\}$ is a finite set of nonzero fractional ideals of R. Then the following are equivalent.*
(i) *$M_R \oplus (\oplus_{j=1}^m K_j)_R$ has a Baer hull.*
(ii) *M_R is semisimple.*
(iii) *$M_R \oplus (\oplus_{j=1}^m K_j)_R$ has a Baer essential extension.*
In this case, $\mathbf{B}(M_R \oplus (\oplus_{j=1}^m K_j)_R) = M_R \oplus (\oplus_{j=1}^m K_j A)_R$, where $A = \sum_{\ell \geq 0} I^{-\ell}$ with $I = Ann_R(M)$. Furthermore, $A = R[q_1, q_2, \dots, q_n]$, where $1 = \sum_{u=1}^n r_u q_u$ with $r_u \in I$ and $q_u \in I^{-1}$, $1 \leq u \leq n$.

Theorem 3.8 ([4, Theorem 1.3]). *Any direct sum of FI-extending modules is FI-extending.*

Theorem 3.9 ([18, Theorem 2.38]). *Assume that R is a commutative domain with the field of fractions F. Let A be an intermediate domain between R and F. Then the following are equivalent.*
(i) *A is a Prüfer domain.*
(ii) *$E(M_R) \oplus A_R^{(n)}$ is the extending hull of $M_R \oplus A_R^{(n)}$ for any R-module M with $Ann_R(M) \neq 0$ and for any positive integer n.*
(iii) *$A_R^{(2)}$ is an extending module.*
(iv) *$A_R^{(2)}$ is a Baer module.*

Example 6. *Let p be a prime integer in $\mathbb{Z}$. Then:*
(i) *$\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ is the extending hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ by Theorem 3.9.*
(ii) *$\mathbb{Z}_p \oplus \mathbb{Z}[1/p]$ is the Baer hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ from Theorem 3.7.*
(iii) *$\mathbb{Z}_{p^2} \oplus \mathbb{Z}$ has the extending hull, which is $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ by Theorem 3.9.*
(iv) *From Theorem 3.7, the $\mathbb{Z}$-module $\mathbb{Z}_{p^2} \oplus \mathbb{Z}$ has no Baer hull because $\mathbb{Z}_{p^2}$ is not semisimple.*
(v) *$\mathbb{Z}_p \oplus \mathbb{Z}$ itself is the FI-extending hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ because $\mathbb{Z}_p \oplus \mathbb{Z}$ is FI-extending as a $\mathbb{Z}$-module by Theorem 3.8.*

There are strong connections between quasi-retractable modules and Baer modules (see Theorem 2.2). Thus for a given module N, one may expect that the smallest quasi-retractable essential extension of N and the smallest Baer essential extension (i.e., Baer hull) of N will coincide if they exist (see Definition 3.3 for the hull of a given module). But the following examples eliminate our expectation.

Example 7. *By Example 1(iii), since $M = \mathbb{Z}_2 \oplus \mathbb{Z}[1/3]$ is quasi-retractable, the smallest quasi-retractable essential extension (i.e., the quasi-retractable hull) of M is M itself. However, the Baer hull of M is $\mathbb{Z}_2 \oplus \mathbb{Z}[1/3][1/2] = \mathbb{Z}_2 \oplus \mathbb{Z}[1/6]$. In fact, by Theorem 3.7,*

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$$

is the Baer hull of $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$. Thus $\mathbb{Z}_2 \oplus \mathbb{Z}[1/6] = \mathbb{Z}_2 \oplus \mathbb{Z}[1/3][1/2]$ is a Baer module by Theorem 3.1. So $\mathbb{Z}_2 \oplus \mathbb{Z}[1/6]$ is the Baer hull of $\mathbb{Z}_2 \oplus \mathbb{Z}[1/3]$ from [13, Theorem 3.6]. Therefore the quasi-retractable hull of M and the Baer hull of M do not coincide.

Example 8. *(i) Let Λ be the set as in Example 1(ii). So $\mathbb{Z}[1/6]^{(\Lambda)}$ is quasi-retractable, but it is not Baer as a $\mathbb{Z}$-module. In this case, by* [13, Theorem 3.6] $\mathbb{Z}[1/6]^{(\Lambda)}$ *has no Baer hull.*
(ii) Let $W = \mathbb{Z}[1/6] \oplus \mathbb{Z}[1/2]$ as a $\mathbb{Z}$-module. Then by [13, Example 3.5(iii)],

$$End(W) = \begin{pmatrix} \mathbb{Z}[1/6] & \mathbb{Z}[1/6] \\ 0 & \mathbb{Z}[1/2] \end{pmatrix}.$$

By the method used in Example 1 (iii) W is quasi-retractable. Note that W is not a Baer module by [13, Example 3.5(iv)]. *Because $T(\mathbb{Z}) = \sum_{\ell \geq 0} \mathbb{Z}^{-\ell} = \mathbb{Z}$ (recall that $T(\mathbb{Z})$ is called the Nagata transform of $\mathbb{Z}$), W has no Baer hull from* [13, Theorem 3.6].

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Sergio Roberto López-Permouth and Nick Pilewski

Modules witnessing that a Leavitt path algebra is directly infinite

Abstract: A ring R is said to be *directly infinite* when there exists a right R-module $B \neq 0$ such that $R \cong R \oplus B$ as a right R-module. In terms of the abelian monoid $V(R)$ of isomorphism classes of finitely generated projective right R-modules, R is directly infinite when there exists a finitely generated projective R-module $B \neq 0$ such that $[R] = [R] + [B]$ in $V(R)$. Given a graph E, we completely identify in terms of the graph E all those finitely generated projective right $L_K(E)$-modules B for which $[L_K(E)] = [L_K(E)] + [B]$ in $V(L_K(E))$.

Keywords: Leavitt path algebra; directly infinite algebras.

1 Introduction

Leavitt path algebras of row finite directed graphs were introduced in [1], and extended to all directed graphs in [2]. Much research on Leavitt path algebras has been concerned with determining graph theoretic conditions on the graph E equivalent to desired algebraic properties of the Leavitt path algebra $L_K(E)$. Among other results, finite dimensional Leavitt path algebras were characterized by their graphs in [3], Leavitt path algebras with the exchange property were characterized and classified by their stable rank in [8], and Leavitt path algebras with zero socle were characterized in [7]. A main result of [5] is the fact that given a row-finite graph E, $V(L_K(E))$, the abelian monoid of isomorphism classes of finitely generated projective left $L_K(E)$-modules, is naturally isomorphic to F, the free abelian monoid on the vertices of E, modulo a congruence on F. Inspired by this congruence, we define and make heavy use of functions on the abelian monoid $\mathbb{N}^{(\mathcal{I})}$ where $\mathcal{I}$ is an indexing set on the vertices of E, albeit our ideas are applied to finitely generated projective *right* $L_K(E)$-modules. This machinery is developed in Section 3.

The notion of an invertible algebra, an R-algebra A with a basis $\mathcal{B}$ of units over R, was introduced in [11]. Papers [10] and [12] deal with determining conditions on a graph E to guarantee that the corresponding Leavitt path algebra $L_K(E)$ has an invertible basis.

This paper is a byproduct of that project since, under certain hypotheses, the condition of invertibility for a Leavitt path algebra $L_K(E)$ is equivalent to E having a sub-

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graph whose corresponding Leavitt path algebra is directly infinite. The search for specific invertible bases led to looking for ideals that witnessed to the direct infinitude of the corresponding Leavitt path algebra. While criteria for a Leavitt path algebra to be directly infinite have been given before ([6, Theorem 3.3]), our contribution here stems in showing an algebraic condition on the incidence matrix that serves the purpose of determining that the algebra is directly infinite while, at the same time, gives a foundation for the construction of the desired pieces of the puzzle. The statement of that necessary property (Proposition 3.1) and the proof of the converse, which culminates with Theorem 3.1, are the bulk of Section 3.

2 Preliminaries

The notion of a Leavitt path algebra was introduced independently in [1] and [5]. A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 of *vertices* and *edges*, respectively, and the *range* and *source* maps $r, s\colon E^1 \to E^0$. For an edge e and vertices u, v, we say that the *source* of e is u or equivalently, u *emits* e if $s(e) = u$, and that v is the *range* of e if $r(e) = v$, or equivalently, v *receives* e. A vertex that emits no edge is called a *sink*, a vertex that receives no edge is called a *source*, and a vertex that is a sink and a source is an *isolated vertex*. A vertex that emits at most finitely (infinitely) many edges is called a *finite (infinite) emitter*. A vertex that is neither a sink nor an infinite emitter is called a *regular vertex*.

If E has no infinite emitters, then we say that E is *row finite*. If E^0 is a finite set and E is row finite, then E^1 must be finite as well, and we simply say that E is a *finite* graph. Given a graph E with $E^0 = \{v_i\colon i \in \mathcal{I}\}$, the *adjacency matrix* $A_E = (a_{ij}) \in \mathbb{N}^{\mathcal{I}\times\mathcal{I}}$ is defined by $a_{ij} = |\{e \in E^1\colon s(e) = v_i, r(e) = v_j\}|$ for all $i, j \in \mathcal{I}$.

Given a subset $S \subseteq E^0$, the *restriction graph* is defined by

$$E_S = (S, \{e \in E^1\colon s(e), r(e) \in S\}, r|_{(E_S)^1}, s|_{(E_S)^1}) .$$

A *path* α of positive length in E is a sequence of edges $\alpha = e_1e_2 \dots e_n$ where $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$ are the *source* and *range* of α, respectively, and n is the *length* of α, denoted $|\alpha|$. An *exit* to the path $\alpha = e_1e_2 \dots e_n$ is an edge f with $s(f) = s(e_i)$ but $f \neq e_i$ for some i. Given $n \in \mathbb{N}$, the set E^n is the set of all paths of length n in E, and $E^* = \bigcup_{n\in\mathbb{N}} E^n$. Note that vertices are considered paths of length 0. A *closed path* is a path $\alpha = e_1, \dots, e_n$ such that $s(e_1) = r(e_n)$. We say that $s(\alpha)$ is the *base* of α. Note that any vertex along a closed path α can be considered a base, after reenumerating the edges of α. A *simple closed path* is a closed path $\alpha = e_1, \dots, e_n$ such that $s(e_i) \neq s(e_1)$ for all $i \neq 1$. A *cycle* is a closed path $\alpha = e_1, \dots, e_n$ such that $s(e_i) \neq s(e_j)$ for all $i \neq j$. Clearly, a cycle is a simple closed path. A *loop* is a cycle of length 1.

Given vertices $u, v \in E^0$, we say that u *connects to* v when there exists a path α in E such that $s(\alpha) = u$ and $r(\alpha) = v$. Note that any vertex v is trivially connected to itself. An undirected graph E is *connected* if, given any two vertices u, v in E, there exists a path from u to v. We will say that a directed graph E is connected when its underlying undirected graph is connected. A graph that is not connected is *disconnected*. A vertex that emits more than one edge is a *bifurcation*, and a vertex that connects to no bifurcation or vertex on a cycle is a *line point*.

Given a graph E, the *extended graph*, or *double graph* of E is the graph $\hat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and r' and s' are defined by

$$r'|_{E^1} = r, s'|_{E^1} = s, r'(e_i^*) = s(e_i), s'(e_i^*) = r(e_i) .$$

Let E be a graph and R be a ring. We say that a collection $\{v, e, e^* : v \in E^0, e \in E^1\}$ is a *Leavitt E-family* in R if the following conditions are satisfied:

1. $v_i v_j = \delta_{ij} v_i$ for all $v_i, v_j \in E^0$
2. $s(e)e = er(e) = e$ for all $e \in E^1$
3. $r(e)e^* = e^* s(e) = e^*$ for all $e^* \in (E^1)^*$
4. (CK1) $e_i^* e_j = \delta_{ij} r(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$
5. (CK2) $v = \sum_{s(e_j)=v} e_j e_j^*$ for every regular vertex v.

Then for a directed graph E and a unital commutative ring R, the *Leavitt path algebra of E with coefficients in* R, or $L_R(E)$, is the universal R-algebra generated by a Leavitt E-family. By "universal,"we mean that for any R-algebra A with a Leavitt E-family $\{a_v, b_e, c_{e^*} : v \in E^0, e \in E^1\}$, there exists an R-algebra homomorphism $\phi\colon L_R(E) \to A$ such that $\phi(v) = a_v$, $\phi(e) = b_e$, and $\phi(e^*) = c_{e^*}$ for all $v \in E^0$ and $e \in E^1$.

A monomial $\alpha\beta^* = \alpha_1, \dots, \alpha_m \beta_n^*, \dots, \beta_1^*$ is said to be *reduced* when $m + n$ is minimal. Observe that if a nonzero monomial $\alpha\beta^* \in L_R(E)$ is of the form $\alpha\beta^* = \alpha' e e^* \beta'^*$ where e is the only edge emitted by α' and β', by the (CK2) relation we have $\alpha\beta^* = \alpha'(ee^*)\beta'^* = \alpha'(r(\alpha)\beta'^* = \alpha'\beta'^*$. In this case, $\alpha\beta^*$ is not a reduced monomial, since it can be rewritten as a monomial $\alpha'\beta'^*$ with $|\alpha'| + |\beta'| < |\alpha| + |\beta|$.

We note that the algebra $L_R(E)$ is spanned over R by the set of paths $\{\alpha\beta^* : r(\alpha) = r(\beta)\}$ in $\hat{E}$. Furthermore, $L_R(E)$ is a $\mathbb{Z}$-graded R-algebra, specifically, $L_R(E) = \bigoplus_{k\in\mathbb{Z}} A_k$ with $A_k = span_R\{\alpha\beta^* \in L_R(E)\colon |\alpha| - |\beta| = k\}$. In addition, $L_R(E)$ is a $\star$-algebra, with linear anti-multiplicative involution $x \mapsto x^*$ defined by $\left(\sum_{i=1}^n k_i \alpha_i \beta_i^*\right)^* = \sum_{i=1}^n k_i \beta_i \alpha_i^*$.

In dealing with the K-theory of Leavitt path algebras, we will employ the conventions used in [5] and [4], and so we now detail that terminology and notation here. Let R be a unital ring and $M_\infty(R)$ be the directed union of $M_n(R)$ as n ranges over $\mathbb{N}$, with transition maps $M_n(R) \to M_{n+1}(R)$ given by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Let $\mathcal{V}(R)$ then be the set of isomorphism classes of finitely generated projective right R-modules. Then $\mathcal{V}(R)$ has a commutative monoid structure, with the operation

$$[P] \oplus [Q] = [P \oplus Q]$$

for isomorphism classes $[P], [Q] \in \mathcal{V}(R)$.

Given a ring R, we will denote the class of finitely generated projective right R-modules by $FP(R)$. Given $B \in FP(R)$, we will denote the direct sum of n copies of B by nB. The abelian monoid of isomorphism classes of $FP(R)$ will be denoted by $V(R)$. Let E be a row-finite graph with vertices $E^0 = \{v_i : i \in \mathcal{I}\}$, and consider the free abelian monoid F on E^0. In [5], the authors take $FP(R)$ to be the class of finitely generated *left* projective R-modules, but speaking instead of finitely generated *right* projective R-modules will not cause problems for us.

Where in [5] elements of F are written in the form $\sum_{i\in\mathcal{I}} v_i$ with the v_i not necessarily distinct, it will be useful for us to write a typical element of F as $\sum_{i\in\mathcal{I}} b_i v_i$ where all $b_i \in \mathbb{N}$ and all v_i are distinct. Furthermore, to be more concise in our proofs we will associate any given $\sum_{i\in\mathcal{I}} b_i v_i \in F$ in the obvious way with an element $\overline{b} = (b_i)_{i\in\mathcal{I}}$ in the abelian monoid $\mathbb{N}^{(\mathcal{I})}$. Note that by elements of this monoid have at most finitely many nonzero b_i. So, every $B \in FP(L_K(E))$ can be associated with an element $\overline{b} \in \mathbb{N}^{(\mathcal{I})}$ by [5, Theorem 3.5], and the remarks following [5, Definition 4.1]. Then from the proof of [4, Theorem 4.3], we can extend this convention from row-finite graphs to graphs that are not row-finite.

3 When $[L_K(E)] \leq [B] \leq [C]$ in $V(L_K(E))$ Implies $[B] = [C]$

Definition 3.1. *Let E be a graph, and let $A = a_{ij}$ be the adjacency matrix of E. Let v_j be a vertex that is a not a sink and let $\overline{b} \in \mathbb{N}^{(\mathcal{I})}$ with $b_j > 0$. Define $f_j : \mathbb{N}^{(\mathcal{I})} \to \mathbb{N}^{(\mathcal{I})}$ by $f_j(b_j) = b_j + a_{jj} - 1$ and $f_j(b_i) = b_i + a_{ji}$ for every $i \neq j$. Given $\overline{b}, \overline{c} \in \mathbb{N}^{(\mathcal{I})}$, we will say that $\overline{b} \to \overline{c}$ in $\mathbb{N}^{(\mathcal{I})}$ when there exist finitely many (not necessarily distinct) vertices $v_{j_1}, \ldots, v_{j_n}$ with $f'_{j_n} \cdots f'_{j_1}(\overline{b}) = \overline{c}$ where $f'_{j_k} = f_{j_k}$ or $f^{-1}_{j_k}$ for every k.*

Remark 3.1. *Given $\overline{b} \in \mathbb{N}^{(\mathcal{I})}$ and vertex v_i, we have $f_i(\overline{b})$ defined if and only if $b_i > 0$. In addition, $f_i^{-1}(\overline{b})$ is defined if and only if $b_j \geq a_{ij}$ for every $j \in \mathcal{I}$.*

Proof. This is clear, from Definition 3.1. □

So, given an element $\overline{b} \in \mathbb{N}^{(\mathcal{I})}$, it is straightforward to determine whether $f_i(\overline{b})$ is defined for a given vertex v_i. However, it is not as easy to determine when a composition of the form $f = f'_{j_n} \cdots f'_{j_1}(\overline{b})$ is defined. Remarks 3.1 and 3.2 will be used often in this section without reference.

Remark 3.2. *Let $\overline{b} \in \mathbb{N}^{(\mathcal{I})}$ and let $f = f'_{j_n} \cdots f'_{j_1} : \mathbb{N}^{(\mathcal{I})} \to \mathbb{N}^{(\mathcal{I})}$ as in Definition 3.1 such that $f(\overline{b})$ is defined with $f(\overline{b}) = \overline{c}$. For every regular vertex $v_i \in E^0$, let $y_i = |\{j_k : j_k = i, f'_{j_k} = f_{j_k}\}|$ and let $z_i = |\{j_k : j_k = i, f'_{j_k} = f^{-1}_{j_k}\}|$. For every sink v_i, let $y_i = z_i = 0$. Finally for every vertex v_i, let $x_i = y_i - z_i$.*

Then for any vertex $v_i \in E^0$, we have

$$\left(\sum_{j \in \mathcal{I}} x_j a_{ji}\right) - x_i = c_i - b_i \ .$$

Proof. By Definition 3.1, for any vertex v_i we have

$$\begin{aligned} c_i = f'_{j_n} \cdots f'_{j_1}(b_i) &= b_i + \left(\sum_{j \in \mathcal{I}} y_j a_{ji}\right) - y_i - \left(\left(\sum_{j \in \mathcal{I}} z_j a_{ji}\right) - z_i\right) \\ &= b_i + \left(\sum_{j \in \mathcal{I}} x_j a_{ji}\right) - x_i \ . \end{aligned}$$ ☐

We now modify [4, Theorem 4.3] and the remarks following [5, Definition 4.1] and write these results in terms of Definition 3.1, which will better suit our needs.

Lemma 3.1. *Let E be a graph with $E^0 = \{v_i : i \in \mathcal{I}\}$ and adjacency matrix $A_E = (a_{ij}) \in \mathbb{N}^{\mathcal{I} \times \mathcal{I}}$. Let $B, C \in FP(L_K(E)) \setminus \{0\}$, and identify them with $\overline{b}, \overline{c} \in \mathbb{N}^{(\mathcal{I})}$, respectively. Then $[B] = [C]$ in $\mathcal{V}(L_K(E))$ if and only if $\overline{b} \to \overline{c}$ in $\mathbb{N}^{(\mathcal{I})}$.*

Proof. Assume first that E is row-finite, and let $\beta = \sum_{i \in \mathcal{I}} b_i v_i$ and $\gamma = \sum_{i \in \mathcal{I}} c_i v_i$ be elements of F associated with B and C, respectively, as in [5, Definition 4.1]. Then by the definition of the congruence $\sim$ on F in [5, Definition 4.1], $\beta \sim \gamma$ if and only if there exists a finite string $\beta = \beta_0, \beta_1, \dots, \beta_n = \gamma$ such that for every $i = 0, \dots, n-1$, either $\beta_i \to_1 \beta_{i+1}$ or $\beta_{i+1} \to_1 \beta_i$ as in [5, Definition 4.1]. This is equivalent to $\overline{b} \to \overline{c}$ in $\mathbb{N}^{(\mathcal{I})}$ as in Definition 3.1. Then by [5, Theorem 3.5] and the remarks following [5, Definition 4.1], this is equivalent to $[B] = [C]$ in $\mathcal{V}(L_K(E))$. Then from the proof of [4, Theorem 4.3], our result is extended to the case where E is not row-finite. ☐

In light of Lemma 3.1, we give a necessary condition of the graph E in order to have two elements of $FP(L_K(E))$ isomorphic as right $L_K(E)$-modules.

Proposition 3.1. *Let E be a graph with $E^0 = \{v_i : i \in \mathcal{I}\}$ and adjacency matrix $A_E = (a_{ij}) \in \mathbb{N}^{\mathcal{I} \times \mathcal{I}}$. Let $B, C \in FP(L_K(E)) \setminus \{0\}$, and identify them with $\overline{b}, \overline{c} \in \mathbb{N}^{(\mathcal{I})}$, respectively. If $[B] = [C]$ in $\mathcal{V}(L_K(E))$, then there exists $x = (x_i) \in \mathbb{Z}^{(\mathcal{I})}$ such that*

- *$x_i = 0$ when v_i is a sink, and*
- *$\left(\sum_{j \in \mathcal{I}} x_j a_{ji}\right) - x_i = c_i - b_i$ for every $i \in \mathcal{I}$.*

Alternatively, define a row vector $y = (y_i) \in \mathbb{Z}^{(\mathcal{I})}$ by $y_i = c_i - b_i$ for $i \in \mathcal{I}$. If $[B] = [C]$ in $\mathcal{V}(L_K(E))$, then there exists a solution $x = (x_i) \in \mathbb{Z}^{(\mathcal{I})}$ to

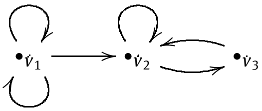
$$x(A_E - I) = y$$

such that $x_i = 0$ when v_i is a sink.

Proof. Assume that $[B] = [C]$ in $\mathcal{V}(L_K(E))$. Then by Lemma 3.1 we have $\overline{b} \to \overline{c}$ in $\mathbb{N}^{(\mathcal{I})}$, hence there exist finitely many vertices $v_{j_1}, \dots, v_{j_n}$ with $f'_{j_n} \cdots f'_{j_1}(\overline{b}) = \overline{c}$ where $f'_{j_k} = f_{j_k}$ or $f^{-1}_{j_k}$ for every k. The conclusion follows from Remark 3.2. ☐

We give the following example to illustrate Proposition 3.1.

Example 1. *Let E be the graph below:*



Let $A = L_K(E)$, and let $B = v_1A \oplus 3v_2A$. First we will show that $[A] = [B]$ in $\mathcal{V}(A)$, and then exhibit the integers x_1, x_2, x_3 as stated in Proposition 3.1. First observe that $A_E = (a_{ij}) = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. As $A = \bigoplus_{i=1}^{3} v_iA$, we can identify A with $(1, 1, 1)$ in $\mathbb{N}^{(\mathcal{I})}$. Then for $f = f_3f_2f_3 : \mathbb{N}^{(\mathcal{I})} \to \mathbb{N}^{(\mathcal{I})}$, we have

$$f((1,1,1)) = f_3f_2f_3((1,1,1)) = f_3f_2((1,2,0)) = f_3((1,2,1)) = (1,3,0)\,,$$

hence $(1,1,1) \to (1,3,0)$ in $\mathbb{N}^{(\mathcal{I})}$, therefore $[L_K(E)] = [B]$ in $\mathcal{V}(L_K(E))$ where $B = v_1L_K(E) \oplus 3v_2L_K(E)$ by Lemma 3.1.

Then as f_1 was not applied, f_2 was applied once, f_3 twice, and f_i^{-1} not applied for any i, we have a solution $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \in \mathbb{Z}^3$ to

$$x(A_E - I) = y$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \end{pmatrix}$$

That is, $\left(\sum_{j=1}^{3} x_ja_{ji}\right) - x_i$ gives the "net change"in the ith projection of $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \in \mathbb{N}^3$ for $1 \le i \le 3$, hence $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 \end{pmatrix}$ in $\mathbb{N}^3$.

Note that for $g = f_3f_1^{-1}f_3f_2f_1$ we have $g((1,1,1)) = (1,3,0)$ as well, so while x_1, x_2, x_3 are unique, our composition f is not. Note also that composition of these functions is not commutative in general: while $f((1,1,1)) = f_3f_2f_3((1,1,1)) = (1,3,0)$, the composition $f'((1,1,1)) = f_2f_3f_3((1,1,1))$ is not defined since $f_3((1,1,1)) = (1,2,0)$. So, the existence of an integral solution $\{x_i : i \in \mathcal{I}\}$ as in the conclusion of Proposition 3.1 does not always clearly imply the existence of a composition $f : \mathbb{N}^{(\mathcal{I})} \to \mathbb{N}^{(\mathcal{I})}$ as in Definition 3.1.

A ring R is said to be *directly infinite* when there exists a right R-module $B \neq 0$ such that $R \cong R \oplus B$ as a right R-module. This condition is equivalent to R not being Dedekind-finite as in [9]. In terms of $\mathcal{V}(R)$, a ring R is directly infinite when there exists a finitely generated projective R-module $B \neq 0$ such that $[R] = [R] + [B]$ in $\mathcal{V}(R)$. Given a graph E with finitely many vertices and $L_K(E)$ directly infinite, we completely identify in terms of the graph E all those finitely generated projective right $L_K(E)$-modules $B = \bigoplus_{i\in\mathcal{I}} b_iv_iL_K(E)$ that witness $L_K(E)$ being directly infinite. We assume E^0 is a finite set, since otherwise $L_K(E)$ itself is not finitely generated as a right $L_K(E)$-module.

A ring R is said to be *separative* when, given $A, B, C \in FP(R)$ with C a direct summand of both nA and nB for some $n \in \mathbb{N}$, $A \oplus C \cong B \oplus C$ implies $A \cong B$. Clearly R is separative if and only if $\mathcal{V}(R)$ is a *separative monoid*, that is, given $A, B, C \in FP(R)$ with $[C] \leq n[A], n[B]$ for some $n \in \mathbb{N}$, $[A] + [C] = [B] + [C]$ in $\mathcal{V}(R)$ implies $[A] = [B]$.

We are now ready to give our main result.

Theorem 3.1. *Let E be a finite graph with $E^0 = \{v_i : i \in \mathcal{I}\}$ and adjacency matrix $A_E = (a_{ij}) \in \mathbb{N}^{\mathcal{I}\times\mathcal{I}}$, and let $B = \bigoplus_{i\in\mathcal{I}} b_i v_i L_K(E)$. Then $[L_K(E)] = [L_K(E)] + [B]$ in $\mathcal{V}(L_K(E))$ if and only if there exist integers $\{x_i : i \in \mathcal{I}\}$ such that*
- *$x_i = 0$ when v_i is a sink, and*
- *$\left(\sum_{j\in\mathcal{I}} x_j a_{ji}\right) - x_i = b_i$ for every $i \in \mathcal{I}$.*

Alternatively, let $b = (b_i) \in \mathbb{N}^{(\mathcal{I})}$. Then $[L_K(E)] = [L_K(E)] + [B]$ in $\mathcal{V}(L_K(E))$ if and only if there exists a solution $x = (x_i) \in \mathbb{Z}^{(\mathcal{I})}$ to

$$x(A_E - I) = b$$

such that $x_i = 0$ when v_i is a sink.

Proof. Let $A = L_K(E)$. Since $A = \bigoplus_{i\in\mathcal{I}} v_i L_K(E)$, we can identify A with $\overline{1} \in \mathbb{N}^{(\mathcal{I})}$, where $1_i = 1$ for all $i \in \mathcal{I}$. So if $[A] = [A] + [B]$ in $\mathcal{V}(A)$, then by Proposition 3.1 there exist integers $\{x_i : i \in \mathcal{I}\}$ such that $x_i = 0$ when v_i is a sink, and $\left(\sum_{j\in\mathcal{I}} x_j a_{ji}\right) - x_i = (1+b_i) - 1 = b_i$ for every $i \in \mathcal{I}$.

Suppose then that there exist integers $\{x_i : i \in \mathcal{I}\}$ as claimed. Given a composition $f = f'_{j_m} \cdots f'_{j_1} : \mathbb{N}^{(\mathcal{I})} \to \mathbb{N}^{(\mathcal{I})}$ as in Definition 3.1 with f_i applied x_i times for every v_i with $x_i > 0$ and f_i^{-1} applied $-x_i$ times for every v_i with $x_i < 0$, our composition has length $m = \sum_{x_i\neq 0} |x_i|$. For every $i \in \mathcal{I}$ such that $x_i \geq 0$, let $s_i = \sum_{x_j<0} x_j a_{ji} - x_i$. For every $i \in \mathcal{I}$ such that $x_i < 0$, let $s_i = \sum_{x_j<0} x_j a_{ji}$. Then since we have only finitely many nonzero x_i, there exists $n := 1 + \max\{-s_i : v_i \in E^0\}$. Define $\overline{n} = (n_i)_{i\in\mathcal{I}} \in \mathbb{N}^{(\mathcal{I})}$ by $n_i = n$ for all $i \in \mathcal{I}$, that is, $\overline{n} \in \mathbb{N}^{(\mathcal{I})}$ corresponds to $nA \in FP(A)$.

Now let $f : \mathbb{N}^{(\mathcal{I})} \to \mathbb{N}^{(\mathcal{I})}$ be a composition $f = f'_{j_m} \cdots f'_{j_1}$ where $|\{j_k : j_k = i, f'_{j_k} = f_{j_k}\}| = x_i$ when $x_i > 0$, and $|\{j_k : j_k = i, f'_{j_k} = f_{j_k}^{-1}\}| = -x_i$ when $x_i < 0$. We claim that $f(\overline{n})$ is defined, and prove this by showing $f'_{j_k} \cdots f'_{j_1}(\overline{n})$ is defined for $1 \leq k \leq m$ by induction on k, where again $m = \sum_{x_i\neq 0} |x_i|$.

First recall from Remark 3.1 that given $v_i \in E^0$ and $\overline{b} \in \mathbb{N}^{(\mathcal{I})}$, we have $f_i(\overline{b})$ defined if and only if $b_i > 0$, and $f_i^{-1}(\overline{b})$ is defined if and only if $b_j \geq a_{ij}$ for all $j \in \mathcal{I}$. For $k = 1$, let $i = j_1$ and assume first that $x_i > 0$. Since $n_i > 0$, $f'_i(\overline{n}) = f_i(\overline{n})$ is defined. Assume now that $x_i < 0$. By our choice of n, for every $v_j \in E^0$ with $a_{ij} \neq 0$ we have $n_j > -\sum_{x_l<0} x_l a_{lj} \geq -x_i a_{ij} \geq a_{ij}$, hence $f'_i(\overline{n}) = f_i^{-1}(\overline{n})$ is defined. So, the base case is proved.

Now assume that $f_{j_k} \cdots f_{j_1}(\overline{n})$ is defined, and let $i = j_{k+1}$. For every $j \in \mathcal{I}$, let $y_j = |\{j_l : 1 \leq l \leq k, j_l = j\}|$ when $x_j \geq 0$, and let $y_j = -|\{j_l : 1 \leq l \leq k, j_l = j\}|$ when $x_j < 0$. So, $|x_j| \geq |y_j|$ for all $j \in \mathcal{I}$, and $|x_i| = |y_i| + 1$. Suppose first that $x_i > 0$, so $f'_i = f_i$. Then by

Remark 3.2, we have

$$
\begin{aligned}
f'_{j_k}\cdots f'_{j_1}(n_i) &= n_i + \left(\sum_{j\in\mathfrak{I}} y_j a_{ji}\right) - y_i \\
&\geq -s_i + \left(\sum_{j\in\mathfrak{I}} y_j a_{ji}\right) - y_i \\
&\geq -\left(\left(\sum_{x_j<0} x_j a_{ji}\right) - x_i\right) + \left(\sum_{j\in\mathfrak{I}} y_j a_{ji}\right) - y_i \\
&\geq \left(\sum_{x_j>0} y_j a_{ji}\right) + 1 > 0\,.
\end{aligned}
$$

So, $f'_i\cdots f'_{j_1}(\overline{n}) = f'_{j_{k+1}}\cdots f'_{j_1}(\overline{n}) = f_{j_{k+1}}f'_{j_k}\cdots f'_{j_1}(\overline{n})$ is defined by Remark 3.1. Suppose now that $x_i < 0$, so $f'_i = f_i^{-1}$. Then for all $v_j \in E^0$ with $a_{ij} \neq 0$ and $x_j > 0$, we have

$$
\begin{aligned}
f'_{j_k}\cdots f'_{j_1}(n_j) &= n_j + \left(\sum_{l\in\mathfrak{I}} y_l a_{lj}\right) - y_j \\
&= n_j + \left(\sum_{l\in\mathfrak{I}} y_l a_{lj}\right) - y_j \\
&\geq -s_j + \left(\sum_{l\in\mathfrak{I}} y_l a_{lj}\right) - y_j \\
&= (-x_i + y_i)a_{ij} + \left(\sum_{x_l<0,\, l\neq i} (-x_l + y_l)a_{lj}\right) + \left(\sum_{x_l>0} y_l a_{lj}\right) + x_j - y_j \\
&\geq a_{ij} + \left(\sum_{x_l>0} y_l a_{lj}\right) \geq a_{ij}\,,
\end{aligned}
$$

and for all $v_j \in E^0$ with $a_{ij} \neq 0$ and $x_j < 0$ we have

$$
\begin{aligned}
f'_{j_k}\cdots f'_{j_1}(n_j) &= n_j + \left(\sum_{l\in\mathfrak{I}} y_l a_{lj}\right) - y_j \\
&= -s_j + \left(\sum_{l\in\mathfrak{I}} y_l a_{lj}\right) - y_j \\
&> -\left(\sum_{x_l<0} x_l a_{lj}\right) + \left(\sum_{l\in\mathfrak{I}} y_l a_{lj}\right) - y_j \\
&= (-x_i + y_i)a_{ij} + \left(\sum_{x_l<0,\, l\neq i} (-x_l + y_l)a_{lj}\right) + \left(\sum_{x_l>0} y_l a_{lj}\right) - y_j \\
&\geq a_{ij} + \left(\sum_{x_l>0} y_l a_{lj}\right) - y_j \geq a_{ij}\,.
\end{aligned}
$$

So, $f_i' \cdots f_{j_1}'(\overline{n}) = f_{j_{k+1}}' \cdots f_{j_1}'(\overline{n}) = f_{j_{k+1}}^{-1} f_{j_k}' \cdots f_{j_1}'(\overline{n})$ is defined by Remark 3.1, and the inductive step is proved.

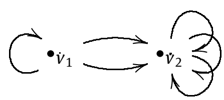
Then by our induction argument, $f(\overline{n}) = f_{j_m}' \cdots f_{j_1}'(\overline{n})$ is defined. Specifically, we have $\overline{c} = (c_i)_{i\in\mathcal{J}} \in \mathbb{N}^{(\mathcal{J})}$ defined by $c_i = f(n_i) = n_i + \left(\sum_{j\in\mathcal{J}} x_j a_{ji}\right) - x_i = n_i + b_i$ for all $i \in \mathcal{J}$. Since $n_i = n \geq 1$ for all $i \in \mathcal{J}$, we have $c_i \geq 1$ for all $i \in \mathcal{J}$. Then since $f(\overline{n}) = \overline{c}$, we have $\overline{n} \to \overline{c}$ in $\mathbb{N}^{(\mathcal{J})}$, hence by Lemma 3.1, $[nA] = [C]$ in $\mathcal{V}(A)$ for $C = \bigoplus_{i\in\mathcal{J}} c_i v_i A$. Since $c_i = n + b_i$ for all $i \in \mathcal{J}$, we have $[nA] = [C] = [nA] + [B]$ in $\mathcal{V}(A)$.

Then since A is separative by [5, Corollary 6.5] and $(n-1)A$ is a direct summand of both $(n-1)A$ and $(n-1)(A \oplus B)$, we have

$$\begin{aligned} [nA] =& [nA] + [B] \\ \Leftrightarrow [A] + [(n-1)A] =& [A] + [B] + [(n-1)A] \\ \Leftrightarrow [A] =& [A] + [B] \end{aligned}$$

in $\mathcal{V}(A)$. □

Example 2. *Let E be the graph below:*



We have $A_E = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right)$, so

$$x(A_E - I) = b$$

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$$

implies $b_1 = 0$ and $b_2 = 2(x_1 + x_2)$. So, $[L_K(E)] = [L_K(E)] + [B]$ in $\mathcal{V}(L_K(E))$ precisely when $[B] = 2n[v_2 L_K(E)]$ for some $n \in \mathbb{N}$. Note that we are not concerned with the values of a solution x_1, x_2 (in fact, there are clearly infinitely many solutions), rather, our point here was to describe the natural numbers b_1, b_2 that imply the existence of a solution.

Of course, not all Leavitt path algebras are directly infinite; an obvious example is the single vertex graph, which is isomorphic to K as a K-algebra.

Corollary 3.1. *Let E be a finite graph with $E^0 = \{v_i : i \in \mathcal{J}\}$ and adjacency matrix $A_E = (a_{ij}) \in \mathbb{N}^{\mathcal{J}\times\mathcal{J}}$. If the existence of $x = (x_i) \in \mathbb{Z}^{(\mathcal{J})}$ such that*
- *$x_i = 0$ when v_i is a sink, and*
- *$\left(\left(\sum_{j\in\mathcal{J}} x_j a_{ji}\right) - x_i\right) \in \mathbb{N}$ for every $i \in \mathcal{J}$*

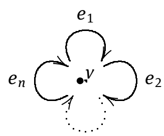
implies $\left(\sum_{j\in\mathcal{J}} x_j a_{ji}\right) - x_i = 0$ for all $i \in \mathcal{J}$, then $L_K(E)$ is not directly infinite. Alternatively, if the existence of a solution $x = (x_i) \in \mathbb{Z}^{(\mathcal{J})}$ to

$$x(A_E - I) = b$$

with $b \in \mathbb{N}^{(\mathcal{J})}$ and $x_i = 0$ when v_i is a sink implies $b = 0$, then $L_K(E)$ is not directly infinite.

Proof. If A_E is such that the hypotheses of the corollary are satisfied, then in terms of Proposition 3.1 we have $b_i = 0$ for all $i \in \mathcal{I}$. Clearly, the only element of $FP(L_K(E))$ that can be identified with $\overline{0} = (0, 0, \ldots) \in \mathbb{N}^{(\mathcal{I})}$ is 0. So by Proposition 3.1, $[L_K(E)] = [L_K(E)] + [B]$ in $\mathcal{V}(L_K(E))$ implies $B = 0$. $\square$

We say that a ring R is of module type $(1, n)$ if $R \cong nR$ as a right R-module with n minimal in this regard. Leavitt path algebras are well known for their ease of construction K-algebras exhibiting this property, a classic example being the Leavitt path algebra $L(1, n)$ of the "rose with n petals" graph:



Another corollary characterizes Leavitt path algebras of module type $(1, n)$ in terms of their graphs.

Corollary 3.2. *Let E be a finite graph with $E^0 = \{v_i : i \in \mathcal{I}\}$ and adjacency matrix $A_E = (a_{ij}) \in \mathbb{N}^{\mathcal{I}\times\mathcal{I}}$, and let $n \in \mathbb{N}^+$. Then $[L_K(E)] = n[L_K(E)]$ in $\mathcal{V}(L_K(E))$ if and only if there exists $x = (x_i) \in \mathbb{Z}^{(\mathcal{I})}$ such that*
- *$x_i = 0$ when v_i is a sink, and*
- *$\left(\sum_{j\in\mathcal{I}} x_j a_{ji}\right) - x_i = n - 1$ for every $i \in \mathcal{I}$.*

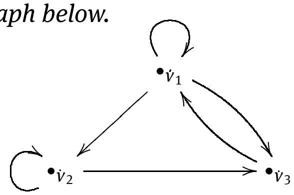
Alternatively, define $b = (b_i) \in \mathbb{Z}^{(\mathcal{I})}$ by $b_i = n - 1$ for all i. Then $[L_K(E)] = n[L_K(E)]$ in $\mathcal{V}(L_K(E))$ if and only if there exists a solution $x = (x_i) \in \mathbb{Z}^{(\mathcal{I})}$ to

$$x(A_E - I) = b$$

such that $x_i = 0$ when v_i is a sink.

Proof. The equation $[L_K(E)] = n[L_K(E)]$ is equivalent to $[L_K(E)] = [L_K(E)] + (n - 1)[L_K(E)]$ in $\mathcal{V}(L_K(E))$, and the conclusion follows directly from Theorem 3.1. $\square$

Example 3. *Let E be the graph below.*



We have $A_E = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$, and

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

has a solution of $x = (1\ 1\ 1)$. So, $L_K(E)$ is of module type $(1, 2)$.

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Arangath Raghavan Rajan

Inductive Groupoids and Normal Categories of Regular Semigroups

Abstract: Inductive groupoids and normal categories arise in the structuretheory of regular semigroups. Inductive groupoid of a regular semigroup S is the groupoid $G(S)\{(x, x') : x'$ is an inverse of $x\}$ which admits certain order structure. Normal category is an abstraction of the structure in the category of principal left [or right] ideals of a regular semigroup with translations as morphisms. Here we show that an inductive groupoid $G(\mathcal{C})$ arises from everynormal category $\mathcal{C}$. In the case when S is a regular semigroup the inductive groupoids $G(S)$ and $G(\mathcal{C})$ where $\mathcal{C}$ is the normal category of principal left ideals of S are closely related. We provide explicit desciption of this relation.

Keywords: Regular semigroup; normal category; groupoid; normal factorization; normal cone.

1 Introduction

Inductive groupoids and normal categories are categories carrying the structure of regular semigroups as shown by Nambooripad (see [6, 7]). The way in which the structure of a regular semigroup is captured in the two objects above are much different. Inductive groupoids directly give rise to semigroups by use of the underlying biordered set of idempotents and certain congruences. But normal categories do not directly assume the biorder structure.The entire structure of the semigroup is determined by the cross connection relation between the categories of principal left ideals and that of principal right ideals (cf.[2, 7]). It is of interest to consider the relation between inductive groupoids and normal categories arising from a regular semigroup. Here we provide a description of the inductive groupoid of a regular semigroup in terms of the groupoids arising from normal categories.

2 Inductive Groupoids

The inductive groupoid associated with a regular semigroup S is the groupoid (cf. [6])

$$G(S) = \{(x, x') : x' \text{ is an inverse of } x\} .$$

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Recall (cf. Mac Lane[5]) that a groupoid is a category in which all morphisms are isomorphisms. For the groupoid $G(S)$ the vertex set is

$$vG(S) = \{(e, e) \colon e \in E(S)\}$$

where $E(S)$ is the set of idempotents of S. For convenience we denote (e, e) by e so that the vertex set of $G(S)$ can be treated as $E(S)$. Then the element $(x, x') \in G(S)$ can be regarded as a morphism from $e_x = xx'$ to $f_x = x'x$. The composition of morphisms is defined as follows.

For $(x, x'), (y, y') \in G(S)$ with $f_x = e_y$ we define

$$(x, x')(y, y') = (xy, y'x') .$$

It is easy to see that $y'x'$ is an inverse of xy whenever $x'x = yy'$. It can be seen that inverse of (x, x') in $G(S)$ is (x', x).

Nambooripad's inductive groupoid is a generalization of Schein's inductive groupoid (cf. [11]). Schein's inductive groupoid generates an inverse semigroup whereas Nambooripad's inductive groupoid generates a regular semigroup. Now $G(S)$ carries the structure of the regular semigroup S through the relation with the underlying biordered set of idempotents of S. Both the above groupoids are essentially ordered groupoids which are defined as follows.

Definition 2.1 ([6]). *A groupoid G with a partial order $\leq$ on G is said to be an ordered groupoid if the following hold.*

(OG1) If $x, y, u, v \in G$, xy, uv exist in G and if $x \leq u$ and $y \leq v$ then $xy \leq uv$.

(OG2) If $x \leq y$ then $x^{-1} \leq y^{-1}$.

(OG3) Let e be an identity in G, $x \in G$ and let e_x be the left identity of x. If $e \leq e_x$ then there exists a unique morphism $e * x \in G$ such that $e * x \leq x$ and the left identity of $e * x$ is e.

Remark 2.1. *The element $e * x$ defined above is often called the restriction of x to e. Dually there is a concept of corestriction defined as follows. Let $x \colon a \to b$ in G and f be an identity in G with $f \leq b$. Then $x * f = (f * x^{-1})^{-1}$ is called the corestriction of x to f. It may be noted that $x * f \leq x$ and the right identity of $(x * f)$ is f.*

3 Normal Categories

Now we proceed to describe normal categories and the associated ordered groupoids. These are categories in which there is a partial order on the set of vertices having some added structures. Ordered categories as described by Lawson [4] are related to these categories. The definition of normal category given here is a modified version of the description given by Nambooripad [7]. We begin with the concept of a category with normal factorization.

Definition 3.1. *A category with normal factorization is a small category $\mathcal{C}$ with the following properties.*

(N1) The vertex set $v\mathcal{C}$ of $\mathcal{C}$ is a partially ordered set such that whenever $a \le b$ in $v\mathcal{C}$, there is a monomorphism $j(a, b)\colon a \to b$ in $\mathcal{C}$. This morphism $j(a, b)$ is called the inclusion from a to b. We consider $(v\mathcal{C}, \le)$ as a category which is a preorder in which a morphism from $a \to b$ is denoted as the pair (a, b) whenenver $a \le b$.

(N2) The map $j\colon (v\mathcal{C}, \le) \to \mathcal{C}$ defined by $(a, b) \mapsto j(a, b)$ is a functor from the preorder $(v\mathcal{C}, \le)$ to $\mathcal{C}$.

(N3) For $a, b, c \in v\mathcal{C}$, with $a, b \le c$, if

$$j(a, c) = fj(b, c)$$

for some $f\colon a \to b$ in $\mathcal{C}$, then $a \le b$ and $f = j(a, b)$.

(N4) Every inclusion $j(a, b)\colon a \to b$ has a right inverse $q\colon b \to a$ such that

$$j(a, b)q = 1_a \,.$$

Such a morphism q is called a retraction in $\mathcal{C}$.

(N5) Every morphism f in $\mathcal{C}$ has a factorization

$$f = quj$$

where q is a retraction, u is an isomorphism and j is an inclusion. Such a factorization is called a normal factorization of f.

Another component of a normal category is normal cone. Normal cones can be defined in a category satisfiying the properties $(N1)$, $(N2)$ and $(N3)$ of the above definition. For convenience we define it on category with normal facorizations.

Definition 3.2. *A normal cone in a category with normal factorization is a map $\gamma\colon v\mathcal{C} \to \mathcal{C}$ satifying the following.*

(i) There is a vertex $c = c_\gamma$ in $v\mathcal{C}$ such that for every $a \in v\mathcal{C}$

$$\gamma(a)\colon a \to c\,.$$

(ii) Whenever $a \le b$, $\gamma(a) = j(a, b)\gamma(b)$.

(iii) There is a vertex $d \in v\mathcal{C}$ such that $\gamma(d)$ is an isomorphism.

Now we give the definition of normal category.

Definition 3.3. *A normal category is a category with normal factorization such that for each $a \in v\mathcal{C}$ there is a normal cone γ with $\gamma(a) = 1_a$.*

The normal category $\mathcal{L}(S)$ of principal left ideals of a regular semigroup S is described as follows.

The vertex set is the set of all principal left ideals of S. That is

$$v\mathcal{L}(S) = \{Se\colon e \in E(S)\}$$

where $E(S)$ is the set of all idempotents of S. We refer to [1] and [3] for details on semigroups. A morphism $\rho\colon Se \to Sf$ is a right translation $x \mapsto xu$ for some $u \in eSf$. We denote this morphism as

$$\rho(e,u,f)\colon Se \to Sf\,.$$

It may be observed that $\rho(e,u,f)\colon e \mapsto u$ and so for $u, u_1 \in eSf$ if $\rho(e,u,f) = \rho(e,u_1,f)$ then $u = u_1$. But $\rho(e,u,f)$ can be equal to $\rho(e',u',f')$ for $e \neq e'$, $u \neq u'$ and $f \neq f'$. We have the following result.

Proposition 3.1. *Let $\rho(e,u,f), \rho(e',u',f') \in \mathcal{L}(S)$,*

$$\rho(e,u,f) = \rho(e',u',f') \textit{ if and only if } e\mathcal{L}e',\ f\mathcal{L}f' \textit{ and } u' = e'u\,.$$

The normal category $\mathcal{R}(S)$ of principal right ideals can be defined similary with left translations as the morphisms.

Now we proceed to describe the ordered groupoid associated with a normal category (cf.[8, 9, 10]). This is the groupoid $G(\mathcal{C})$ of isomorphisms in the normal category $\mathcal{C}$. Since for each $a \in v\mathcal{C}$, $1_a\colon a \to a$ is an isomorphism we have $v\mathcal{C} = vG(\mathcal{C})$. So $vG(\mathcal{C})$ has a partial order arising from $\mathcal{C}$. In the next theorem we extend this partial order to $G(\mathcal{C})$. First we have a proposition.

Proposition 3.2. *Let $\mathcal{C}$ be a normal category and $f\colon a \to b$ be a monomorphism in $\mathcal{C}$. Then if $f = quj$ is a normal factorization of f then $q = 1$.*

Proof. Let

$$f = quj$$

where $q\colon a \to a_0$, $u\colon a_0 \to b_0$ and $j\colon b_0 \to b$ for $a_0 \le a$ and $b_0 \le b$. Now

$$j(a_0,a)f = j(a_0,a)quj = uj\,.$$

So

$$qj(a_0,a)f = quj = f = 1_a f\,.$$

Since f is a monomorphism we can cancel on the right and so

$$qj(a_0,a) = 1_a\,.$$

Now by (N3) of the definition of normal category we have q is an inclusion and so $a \le a_0$. Thus $a = a_0$ and $q = 1_a$. □

The following theorem describes restrictions of isomorphisms and this will lead to an extension of the prtial order. In the following the identity morphism 1_a will be denoted also by a so that any $a \in v\mathcal{C}$ will be treated as an element of $G(\mathcal{C})$.

Theorem 3.1. *Let $G(\mathcal{C})$ be the groupoid of isomorphisms of a normal category $\mathcal{C}$. Let $u\colon b \to c$ in $G(\mathcal{C})$ and $a \le b$ in $v\mathcal{C}$. Then there is a unique isomorphism u_1 such that*

$$j(a,b)u = u_1 j_1$$

for an inclusion j_1.

Proof. By Proposition 3.2 we see that since $j(a, b)u$ is a monomorphism,

$$j(a, b)u = u_1 j_1$$

for an isomorphism $u_1 : a \to c_0$ and an inclusion $j_1 : c_0 \to c$ where $c_0 \le c$. It remains to prove the uniqueness of u_1. Let

$$j(a, b)u = u_1 j_1 = u_2 j_2$$

for an isomorphism $u_2 : a \to c_2$ and an inclusion $j_2 : c_2 \to c$ where $c_2 \le c$. Now

$$j_1 = u_1^{-1} u_2 j_2 \text{ and } j_2 = u_2^{-1} u_1 j_1 \ .$$

It follows by (N3) of normal category that $u_1^{-1} u_2$ and $u_2^{-1} u_1$ are inclusions and so $c_0 = c_2$ and so $u_1 = u_2$. ☐

Now we give the theorem describing the partial order on $G(\mathcal{C})$ making it an ordered groupoid.

Theorem 3.2. *Let $\mathcal{C}$ be a normal category. Define partial order on $G(\mathcal{C})$ as follows. For $x : a \to b$ and $y : c \to d$ in $G(\mathcal{C})$,*

$$x \le y \text{ if } a \le c,\ b \le d \text{ and } xj(b, d) = j(a, c)y \ .$$

Then $(G(\mathcal{C}), \le)$ is an ordered groupoid.

Proof. Clearly $\le$ is a partial order and (OG1) and (OG2) are easily verified. Towards proving (OG3) consider an identity a in $G(\mathcal{C})$ and $x : b \to c$ in $G(\mathcal{C})$ with $a \le b$. Define

$$a * x = x_1$$

where x_1 is the unique isomorphism for which $j(a, b)x = x_1 j$ given by Proposition 3.1. Clearly $x_1 \le x$ and left identity of x_1 is a. Hence axiom $(OG3)$ is satisfied and thus $G(\mathcal{C})$ is an ordered groupoid. ☐

4 The Inductive Groupoid $G(S)$ and the Ordered Groupoid $G(\mathcal{L}(S))$

Here we describe the ordered groupoid $G(\mathcal{L}(S))$ correponding to the normal category $\mathcal{L}(S)$ and obtain the relations between this groupoid and the inductive groupoid $G(S)$ of the semigroup S. The following theorem gives a homomorphism between these ordered groupoids.

Theorem 4.1. *Let $(G(S) = \{(x, x') : x'$ is an inverse of $x\}$ be the inductive groupoid of a regular semigroup S. Let $e_x = xx'$ and $f_x = x'x$. For every $(x, x') \in G(S)$ let $\rho(x, x') = \rho(e_x, x, f_x) : Se_x \to Sf_x$ be a morphism in $\mathcal{L}(S)$. Then $\rho(e_x, x, f_x)$ is an isomorphism in $\mathcal{L}(S)$ and*

$$\phi : G(S) \to G(\mathcal{L}(S)) \text{ given by } (x, x') \mapsto \rho(e_x, x, f_x)$$

is a homomorphism of ordered groupoids.

Proof. Clearly $\rho(e_x, x, f_x)$ is an isomorphism in $\mathcal{L}(S)$ with $\rho(f_x, x', e_x)$ as the inverse. For

$$\rho(e_x, x, f_x)\rho(f_x, x', e_x) = \rho(e_x, xx', e_x) = \rho(e_x, e_x, e_x)$$

which is identity on Se_x. Similarly $\rho(f'_x, x, e_x)\rho(e_x, x, f_x)$ is identity on Sf_x.

To show that ϕ is a homomorphism consider $(x, x'), (y, y') \in G(S)$ with $f_x = x'x = yy' = e_y$. Then $(x, x')(y, y') = (xy, y'x') \in G(S)$. Now

$$\begin{aligned}\phi((x, x')(y, y')) &= \phi(xy, y'x')\\ &= \rho(e_x, xy, f_y) \text{ since } e_{xy} = e_x\,, \text{ and } f_{xy} = f_y\,.\\ &= \rho(e_x, x, f_x)\rho(f_x, y, f_y)\\ &= \phi(x, x')\phi(y, y')\,.\end{aligned}$$

Now we show that ϕ preserves the partial order on the groupoids. Towards this consider $(x, x'), (y, y') \in G(S)$ with $(x, x') \le (y, y')$. Then by the definition of partial order in the groupoid $G(S)$ (cf. [6]) we have $e_x \le e_y$, $f_x \le f_y$ and

$$(x, x') = e_x * (y, y') = (e_x y, y'e_x)\,.$$

Now $\phi(x, x') = \rho(e_x, x, f_x)$ and $\phi(y, y') = \rho(e_y, y, f_y)$ and so

$$\begin{aligned}j(e_x, e_y)\phi(y, y') &= j(e_x, e_y)\rho(e_y, y, f_y)\\ &= \rho(e_x, e_x y, f_y)\\ &= \rho(e_x, x, f_y)\\ &= \rho(e_x, x, f_x)\rho(f_x, f_x, f_y)\\ &= \phi(x, x')j(f_x, f_y)\,.\end{aligned}$$

So $\phi(x, x') \le \phi(y, y')$. Thus ϕ is a homomorphism of ordered groupoids. ☐

Now we proceed to describe the kernel of this homomorphism. Let $\cong$ denote the kernel of ϕ. Sometimes we denote the kernel of ϕ by $\cong_L$ to indicate that it is arising from the normal category $\mathcal{L}(S)$. We have the following theorem.

Theorem 4.2. *Let $(x, x'), (y, y') \in G(S)$. Then $(x, x') \cong (y, y')$ if and only if*
(i) $x\mathcal{L}y$ and $x'\mathcal{L}y'$ and
(ii) $x \perp y'$ and $y \perp x'$ where $\perp$ denotes inverse.

Proof. Suppose that $(x, x') \cong (y, y')$. Then $\rho(e_x, x, f_x) = \rho(e_y, y, f_y)$. So

$$e_x\mathcal{L}e_y,\ f_x\mathcal{L}f_y,\ e_x y = x \text{ and } e_y x = y\,.$$

So

$$\begin{aligned}xy'x &= x(y'e_y)x\\ &= xy'(e_y x)\\ &= xy'y\\ &= xf_y\\ &= x\,.\end{aligned}$$

Similarly

$$y'xy' = y'e_xyy' = y'yy' = y' \ .$$

So $x \perp y'$. Similarly we can see that $y \perp x'$.

Now we prove the converse. Suppose the conditions (i) and (ii) hold. Then $\rho(e_x, x, f_x)$, $\rho(e_y, y, f_y)\colon Se \to Sf$ where $e\mathcal{L}e_x\mathcal{L}e_y$ and $f\mathcal{L}f_x\mathcal{L}f_y$. To show that $\rho(e_x, x, f_x) = \rho(e_y, y, f_y)$, it is sufficient to prove that $e_xy = x$. Now

$$e_xy = xx'y = xx'x = x \text{ since } x'y = x'x \text{ from } (i) \text{ and } (ii) \ .$$

This proves the theorem. □

Similarly we have a homorphism $\psi\colon G(S) \to \mathcal{R}(S)$ where $\mathcal{R}(S)$ is the normal category of principal right ideals of S with left translations as the morphisms. Here ψ is defined by

$$\psi(x, x') = \lambda(e_x, x', f_x)\colon e_xS \to f_xS \ .$$

The following theorem charcaterizes the kernel of ψ. We denote kerψ by $\cong_R$.

Theorem 4.3. *Let $(x, x'), (y, y') \in G(S)$. Then $(x, x') \cong_R (y, y')$ if and only if*

(i) $x\mathcal{R}y$ and $x'\mathcal{R}y'$ and

(ii) $x \perp y'$ and $y \perp x'$.

Remark 4.1. *It is easy to see that for $(x, x'), (y, y') \in G(S)$ if $(x, x') \cong_L (y, y')$ and $(x, x') \cong_R (y, y')$ then $(x, x') = (y, y')$.*

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Nadeem ur Rehman, Mohd Arif Raza, and Sajad Ahmad Pary

Actions of generalized derivations in Rings and Banach Algebras

Abstract: We investigate the action of generalized derivation δ associated with a derivation α in a prime ring $\mathcal{R}$ satisfying (i) $\delta(u)^m - (u)^n \in Z(\mathcal{R})$, for all $u \in \mathcal{L}$, a non-central Lie ideal of $\mathcal{R}$ (ii) $[\delta(x), \alpha(y)]_m = \delta([x, y])^n$, for all $x, y \in \mathcal{I}$, a nonzero ideal of $\mathcal{R}$, where m, n are fixed positive integers. Moreover, we also examine the case when $\mathcal{R}$ is semiprime ring. Finally as an application, we obtain some range inclusion results of continuous generalized derivations on Banach algebra $\mathfrak{A}$.

Keywords: Banach algebra; Generalized derivation; Martindale ring of quotient; Prime and semiprime ring; Radical.

1 Introduction, Notation, and Statements of the Results

Daif and Bell [8] discussed the actions of derivation in semiprime rings. More precisely, they showed that in a semiprime ring $\mathcal{R}$, there exists a nonzero ideal $\mathcal{I}$ of $\mathcal{R}$ and a derivation α of $\mathcal{R}$ such that $\alpha([x, y]) = [x, y]$, for all $x, y \in \mathcal{I}$, then $\mathcal{I} \subseteq Z(\mathcal{R})$. Later, Quadri et al. [31] proved that if $\mathcal{R}$ is a prime ring, $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$ and δ is a generalized derivation associated with a nonzero derivation α of $\mathcal{R}$ such that $\delta([x, y]) = [x, y]$, for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative. Recently, Huang and Davvaz [15] proved that if $\mathcal{R}$ is a prime ring and $\mathcal{R}$ admits a generalized derivation δ associated with a nonzero derivation α such that $\delta([x, y])^m = [x, y]^n$, for all $x, y \in \mathcal{R}$, where m, n are fixed positive integers, then $\mathcal{R}$ is commutative.

In 1994, Bell and Daif [3] initiated the study of strong commutativity preserving (scp) maps (see [24, 32, 33, 34] and references therein) and proved that a nonzero right ideal $\mathcal{I}$ of a semiprime ring $\mathcal{R}$ is central if $\mathcal{R}$ admits a derivation which is scp on $\mathcal{I}$. More precisely, they proved that if $\mathcal{R}$ is a semiprime ring, $\mathcal{I}$ is a nonzero right ideal of $\mathcal{R}$ and $\mathcal{R}$ admits a derivation α such that $[x, y] = [\alpha(x), \alpha(y)]$, for all $x, y \in \mathcal{I}$, then $\mathcal{I} \subseteq Z(\mathcal{R})$. Motivated by the above result, Huang [14] obtained that if $\mathcal{R}$ is a prime ring with $char(\mathcal{R}) \neq 2$, $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$ and α is a nonzero derivation of $\mathcal{R}$ such that $[\alpha(x), \alpha(y)]_m = [x, y]^n$, for all $x, y \in \mathcal{I}$, where m, n are fixed positive in-

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tegers, then $\mathcal{R}$ is commutative. Very recently, Raza and Rehman [33] studied a similar result for generalized derivation in prime and semiprime rings and obtained the same conclusion. Inspired by the previous results, in this manuscript, we study prime and semiprime rings admitting a generalized derivation δ satisfying the following condition

1. $\delta(u)^m - (u)^n \in Z(\mathcal{R})$, for all $u \in \mathcal{L}$;
2. $[\delta(x), \alpha(y)]_m = \delta([x, y])^n$, for all $x, y \in \mathcal{I}$.

Finally by using the above identity, we obtain some results of continuous bounded generalized derivations on Banach algebra.

Throughout this paper, unless otherwise stated, $\mathcal{R}$ is a (semi)prime ring, $Z(\mathcal{R})$ is the center of $\mathcal{R}$, $\mathcal{Q}$ is the Martindale quotient ring of $\mathcal{R}$ and $\mathcal{U}$ is the Utumi quotient ring of $\mathcal{R}$. The center of $\mathcal{U}$, denoted by $\mathcal{C}$, is called the extended centroid of $\mathcal{R}$ (we refer the reader to [2], for the definitions and related properties of these objects). For $x, y \in \mathcal{R}$ and each $n \geq 0$, set $[x, y]_0 = x$, $[x, y]_1 = xy - yx$, then an Engel polynomial is a polynomial $[x, y]_n = [[x, y]_{n-1}, y]$, $n = 1, 2, \cdots$, in non-commuting indeterminates x and y. The ring $\mathcal{R}$ satisfies an Engel condition if, there exists a positive integer n such that $[x, y]_n = 0$. Recall that a ring $\mathcal{R}$ is prime if, for any $\mathrm{a}, \mathrm{b} \in \mathcal{R}$, $\mathrm{a}\mathcal{R}\mathrm{b} = (0)$ implies $\mathrm{a} = 0$ or $\mathrm{b} = 0$, and is semiprime if, for any $\mathrm{a} \in \mathcal{R}$, $\mathrm{a}\mathcal{R}\mathrm{a} = (0)$ implies $\mathrm{a} = 0$. An additive subgroup $\mathcal{L}$ of $\mathcal{R}$ is said to be a Lie ideal if $[l, \mathrm{r}] \in \mathcal{L}$, for all $l \in \mathcal{L}$ and $\mathrm{r} \in \mathcal{R}$. A Lie ideal $\mathcal{L}$ is said to be non-commutative if $[\mathcal{L}, \mathcal{L}] \neq 0$. Let $\mathcal{L}$ be a non-commutative Lie ideal of $\mathcal{R}$. Then it is well known that $[\mathcal{R}[\mathcal{L}, \mathcal{L}]\mathcal{R}, \mathcal{R}] \subseteq \mathcal{L}$. Since $[\mathcal{L}, \mathcal{L}] \neq 0$, we find that $0 \neq [\mathcal{I}, \mathcal{R}] \subseteq \mathcal{L}$, for $\mathcal{I} = \mathcal{R}[\mathcal{L}, \mathcal{L}]\mathcal{R}$, a nonzero ideal of $\mathcal{R}$. An additive mapping $\alpha\colon \mathcal{R} \to \mathcal{R}$ is called a derivation if $\alpha(xy) = \alpha(x)y + x\alpha(y)$ holds, for all $x, y \in \mathcal{R}$. In particular, α is an inner derivation induced by an element $\mathrm{a} \in \mathcal{R}$, if $\alpha(x) = \mathcal{I}_{\mathrm{a}}(x) = [\mathrm{a}, x]$, for all $x \in \mathcal{R}$. Many results in the literature indicate that the global structure of a ring $\mathcal{R}$ is often tightly connected to the behaviour of additive mappings defined on $\mathcal{R}$. By a generalized inner derivation on $\mathcal{R}$, one usually means an additive mapping $\delta\colon \mathcal{R} \to \mathcal{R}$ such that $\delta(x) = \mathrm{a}x + x\mathrm{b}$, for fixed $\mathrm{a}, \mathrm{b} \in \mathcal{R}$. For such a mapping δ, it is easy to see that $\delta(xy) = \delta(x)y + x[y, \mathrm{b}] = \delta(x)y + x\mathcal{I}_{\mathrm{b}}(y)$, where $\mathcal{I}_{\mathrm{b}}$ is an inner derivation determined by b. This observation leads to the definition given in [5]: an additive mapping $\delta\colon \mathcal{R} \to \mathcal{R}$ is called generalized derivation associated with a derivation α if $\delta(xy) = \delta(x)y + x\alpha(y)$, for all $x, y \in \mathcal{R}$. Familiar examples of generalized derivations are derivations and generalized inner derivations. Since the sum of two generalized derivations is a generalized derivation, every map of the form $\delta(x) = \mathrm{c}x + \alpha(x)$ is a generalized derivation, where c is a fixed element of $\mathcal{R}$ and α is a derivation of $\mathcal{R}$.

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations. The Engel type identity with derivation appeared in the well-known paper of Posner [30] who proved that a prime ring $\mathcal{R}$ admitting a nonzero de-

rivation α of $\mathcal{R}$ such that $[\alpha(x), x] \in Z(\mathcal{R})$, for all $x \in \mathcal{R}$, must be commutative. Since then, several authors have studied this kind of Engel type identities with derivations and generalized derivations in different ways. In [23], Lee extended the definition of generalized derivation in the following sense: an additive mapping $\delta\colon \mathcal{J} \to \mathcal{U}$, from a dense right ideal $\mathcal{J}$ of $\mathcal{R}$ into $\mathcal{U}$, is called a generalized derivation if there exists a derivation $\alpha\colon \mathcal{J} \to \mathcal{U}$ such that $\delta(\mathrm{xy}) = \delta(x)y + x\alpha(y)$ for all $x, y \in \mathcal{J}$. He proved that every generalized derivation can be uniquely extended to a generalized derivation of $\mathcal{U}$ and it assumes the form $\delta\colon x \mapsto \mathrm{a}x + \alpha(x)$ for some $\mathrm{a} \in \mathcal{U}$ and a derivation α on $\mathcal{U}$. More related results about generalized derivations can be found in [9, 10, 16, 31, 33] and references therein.

We are now ready to state the main results:

Theorem 1.1. *Let $\mathcal{R}$ be a prime ring with center $Z(\mathcal{R})$, C be the extended centroid of $\mathcal{R}$ and $\mathcal{L}$ be a non-central Lie ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation δ associated with a nonzero derivation α such that $\delta(u)^m - (u)^n \in Z(\mathcal{R})$ for all $u \in \mathcal{L}$, where m, n are fixed positive integers, then $dim_{\mathcal{C}}\mathcal{RC} = 4$.*

Theorem 1.2. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2 and $\mathcal{J}$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation δ associated with a nonzero derivation α such that $[\delta(x), \alpha(y)]_m = \delta([x, y])^n$, for all $x, y \in \mathcal{J}$, where m, n are fixed positive integers, then $\mathcal{R}$ is commutative.*

Theorem 1.3. *Let $\mathcal{R}$ be a semiprime ring of characteristic different from 2 with center $Z(\mathcal{R})$. If $\mathcal{R}$ admits a generalized derivation δ associated with a nonzero derivation α such that $[\delta(x), \alpha(y)]_m = \delta([x, y])^n$, for all $x, y \in \mathcal{R}$, where m, n are fixed positive integers, then there exists a central idempotent element e in $\mathcal{U}$ such that on the direct sum decomposition $\mathcal{U} = \mathrm{e}\mathcal{U} \oplus (1 - \mathrm{e})\mathcal{U}$, α vanishes identically on $\mathrm{e}\mathcal{U}$ and the ring $(1 - \mathrm{e})\mathcal{U}$ is commutative.*

In the last section of this paper, we will consider $\mathfrak{A}$ as a Banach algebra with Jacobson radical $rad(\mathfrak{A})$. The classical result of Singer and Wermer [36] stated that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. Singer and Wermer also formulated the conjecture that the continuity assumption can be removed. In 1988, Thomas [37] verified this conjecture. It is clear that the same result of Singer and Wermer does not hold in non-commutative Banach algebras (because of inner derivations). However, this situation raises a very interesting question as how to obtain the non-commutative version of the Singer-Wermer theorem. A first answer to this problem was obtained by Sinclair [35]. He proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. Since then, many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebra.

In [26], Mathieu and Murphy proved that if α is a continuous derivation on an arbitrary Banach algebra such that $[\alpha(x), x] \in Z(\mathfrak{A})$, for all $x \in \mathfrak{A}$, then α maps into the radical. Later, Mathieu and Runde [27] removed the continuity assumption us-

ing the classical result of Posner's [30] on centralizing derivations of prime rings and Thomas's theorem [37] in which they showed that if α is a derivation of $\mathfrak{A}$ such that $[\alpha(x), x] \in Z(\mathfrak{A})$, for all $x \in \mathfrak{A}$, then α has its range in the radical of the algebra. More recently, Park [29] proved that if α is a derivation of a non-commutative Banach algebra $\mathfrak{A}$ such that $[[\alpha(x), x], \alpha(x)] \in rad(\mathfrak{A})$, for all $x \in \mathfrak{A}$, then α maps into $rad(\mathfrak{A})$. In [9], De Filippis extended the Park's result to the generalized derivation.

Here, we will continue the investigation about the relationship between the structure of an algebra $\mathfrak{A}$ and the behaviour of generalized derivations defined on $\mathfrak{A}$. After that, we apply our first result on prime ring to study the analogous conditions for continuous generalized derivations on Banach algebra.

More precisely, we will prove the following:

Theorem 1.4. *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and m, n be the fixed positive integers. Suppose $\delta = L_{\mathrm{a}} + \alpha$ is a continuous generalized derivation of $\mathfrak{A}$ for some element $\mathrm{a} \in \mathfrak{A}$ and α is a derivation of $\mathfrak{A}$. If $[\delta(x), \alpha(y)]_m - \delta([x, y])^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, then $\alpha(\mathfrak{A}) \subseteq rad(\mathfrak{A})$.*

2 The results in Prime Rings

For the proof of our main results, we need the following facts, which might be of some independent interest.

Fact 2.1 ([6]). *If $\mathfrak{I}$ is a two-sided ideal of $\mathfrak{R}$, then $\mathfrak{R}$, $\mathfrak{I}$ and $\mathfrak{U}$ satisfy the same generalized polynomial identities with coefficient in $\mathfrak{U}$.*

Fact 2.2 ([2, Proposition 2.5.1]). *Every derivation α of $\mathfrak{R}$ can be uniquely extended to a derivation of $\mathfrak{U}$.*

Fact 2.3 ([19]). *Let $\mathfrak{R}$ be a prime ring, α be a nonzero derivation of $\mathfrak{R}$ and $\mathfrak{I}$ be a nonzero two-sided ideal of $\mathfrak{R}$. If $f(x_1, \cdots, x_n, \alpha(x_1), \cdots, \alpha(x_n))$ is a differential identity in $\mathfrak{I}$, i.e.,*

$$f(\mathrm{r}_1, \cdots, \mathrm{r}_n, \alpha(\mathrm{r}_1), \cdots, \alpha(\mathrm{r}_n)) = 0, \text{ for all } \mathrm{r}_1, \cdots, \mathrm{r}_n \in \mathfrak{I},$$

then one of the following holds:

1. *α is an inner derivation in $\mathfrak{Q}$, the Martindale quotient ring of $\mathfrak{R}$, in the sense that there exists $\mathrm{q} \in \mathfrak{Q}$ such that $\alpha(x) = [\mathrm{q}, x]$, for all $x \in \mathfrak{R}$, and $\mathfrak{I}$ satisfies the generalized polynomial identity*

$$f(\mathrm{r}_1, \cdots, \mathrm{r}_n, [\mathrm{q}, \mathrm{r}_1], \cdots, [\mathrm{q}, \mathrm{r}_n]) = 0\,;$$

2. *$\mathfrak{I}$ satisfies the generalized polynomial identity*

$$f(x_1, \cdots, x_n, y_1, \cdots, y_n) = 0\,.$$

Fact 2.4. *Let $\mathcal{R}$ be a prime ring and $\mathcal{L}$ be a non-central Lie ideal of $\mathcal{R}$. Then either $char(R) = 2$ and $dim_{\mathcal{C}}\mathcal{RC} = 4$ or there exists a non-central two-sided ideal $\mathcal{I}$ of $\mathcal{R}$ such that $0 \neq [\mathcal{I}, \mathcal{R}] \subseteq \mathcal{L}$.*

Proof. If $char(\mathcal{R}) \neq 2$, then this result is contained in Lemma 2 of [4]. In case $char(\mathcal{R}) = 2$, it follows from Theorem 4 of [21] and Lemma 2 of [11]. □

The following fact is implicitly contained in Theorem 4 of [13].

Fact 2.5. *Let $\mathcal{R}$ be a non-commutative prime ring and $m \geq 1$ be a fixed integer such that $[x, y]^m \in Z(\mathcal{R})$, for all $x, y \in \mathcal{R}$. Then $dim_{\mathcal{C}}\mathcal{RC} = 4$.*

Fact 2.6 ([1, Lemma 7.1]). *Let ${}_{\mathcal{D}}\mathcal{M}$ be a left vector space over a division ring $\mathcal{D}$ with $dim_{\mathcal{D}}\mathcal{M} \geq 2$ and $\mathcal{T} \in End(\mathcal{M})$. If x and xT are $\mathcal{D}$-dependent for every $x \in \mathcal{M}$, then there exists $\lambda \in \mathcal{D}$ such that $xT = \lambda x$ for all $x \in \mathcal{M}$.*

We begin with the following lemma:

Lemma 2.1. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2, $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$ and m, n be the fixed positive integers. If $\mathcal{R}$ admits a nonzero derivation α such that $[\alpha(x), \alpha(y)]_m = (\alpha([x, y]))^n$, for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative.*

Proof. By given hypothesis, we have

$$\begin{aligned}
[\alpha(x), \alpha(y)]_m &= (\alpha([x, y]))^n \\
&= ([\alpha(x), y] + [x, \alpha(y)])^n, \text{ for all } x, y \in \mathcal{I}.
\end{aligned}$$

In the light of Kharchenko's theory [19], we divide the proof into two cases:
Case 1. If α is $\mathcal{Q}$-outer, then I satisfies the polynomial identity

$$[s, t]_m = ([s, y] + [x, t])^n, \text{ for all } x, y, s, t \in \mathcal{I}.$$

In particular, for $s = 0$, $\mathcal{I}$ satisfies the blended component $[x, t]^n = 0$, for all $x, t \in \mathcal{I}$, and hence $\mathcal{R}$ is commutative by Herstein [13, Theorem 2].
Case 2. If α is $\mathcal{Q}$-inner induced by an element $q \in \mathcal{Q}$, i.e., $\alpha(x) = [q, x]$, for all $x \in \mathcal{R}$,

then we have $[[q, x], [q, y]]_m = ([[q, x], y] + [x, [q, y]])^n$, for all $x, y \in \mathcal{I}$. By Chuang [6, Theorem 1], $\mathcal{I}$ and $\mathcal{Q}$ satisfy same generalized polynomial identities (GPIs), i.e., $[[q, x], [q, y]]_m = ([[q, x], y] + [x, [q, y]])^n$, for all $x, y \in \mathcal{Q}$. If the center $\mathcal{C}$ of $\mathcal{Q}$ is infinite, then we have $[[q, x], [q, y]]_m = ([[q, x], y] + [x, [q, y]])^n$, for all $x, y \in \mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is algebraic closure of $\mathcal{C}$. Since both $\mathcal{Q}$ and $\mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ are prime and centrally closed [12, Theorem 2.5 and Theorem 3.5], we may replace $\mathcal{R}$ by $\mathcal{Q}$ or $\mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ according as $\mathcal{C}$ is finite or infinite. Thus, we may assume that $\mathcal{R}$ is centrally closed over $\mathcal{C}$ (*i.e.*, $\mathcal{RC} = \mathcal{R}$) which is either finite or algebraically closed and $[[q, x], [q, y]]_m = ([[q, x], y] + [x, [q, y]])^n$, for all $x, y \in \mathcal{R}$. By Martindale [25, Theorem 3], $\mathcal{RC}$ (and so $\mathcal{R}$) is a primitive ring having nonzero socle $\mathcal{H}$ with $\mathcal{C}$ as the associated division ring. Hence, by Jacobson's theorem

[17, p.75], $\mathcal{R}$ is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{C}$ and $\mathcal{H}$ consists of the finite rank linear transformations in $\mathcal{R}$. If $\mathcal{V}$ is finite dimensional over $\mathcal{C}$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong \mathcal{M}_k(\mathcal{C})$, where $k = dim_{\mathcal{C}}\mathcal{V}$.

Suppose that $dim_{\mathcal{C}}\mathcal{V} \geq 2$, otherwise we have done. Now, we want to show that v and qv are linearly $\mathcal{C}$-dependent, for all $v \in \mathcal{V}$. If $qv = 0$, then $\{v, qv\}$ is linearly $\mathcal{C}$-dependent. Suppose on contrary that v and qv are linearly $\mathcal{C}$-independent, for some $v \in \mathcal{C}$.

If $q^2v \notin Span_{\mathcal{C}}\{v, qv\}$, then the set $\{v, qv, q^2v\}$ is linearly $\mathcal{C}$-independent. Since v and qv are linearly $\mathcal{C}$-independent, by the density of $\mathcal{R}$, there exist $x_0, y_0 \in \mathcal{R}$ such that

$$x_0v = v, \quad x_0(qv) = 0, \quad x_0(q^2v) = 0\,;$$
$$y_0v = 0, \quad y_0(qv) = -v, \quad y_0(q^2v) = 0\,.$$

Thus, $2^mqv - v = ([[q, x_0], [q, y_0]]_m)v - ([q, [x_0, y_0]]^n)v = 0$, and this implies $2^mqv = v$, a contradiction.

If $q^2v \in Span_{\mathcal{C}}\{v, qv\}$, then $q^2v = v\beta + qv\gamma$, for some $\beta, \gamma \in \mathcal{C}$. By the density of $\mathcal{R}$, there exist $x_0, y_0 \in \mathcal{R}$ such that

$$x_0v = v, \quad x_0(qv) = 0\,;$$
$$y_0v = 0, \quad y_0(qv) = -v\,.$$

Therefore, $2^mqv - 2^{m-1}v\gamma - v = ([[q, x_0], [q, y_0]]_m - ([q, [x_0, y_0])^n)v = 0$ and hence $2^mqv = (2^{m-1}\gamma + 1)v$, a contradiction. So, we conclude that $\{v, qv\}$ is linearly $\mathcal{C}$-dependent, for all $v \in \mathcal{V}$. Thus, by Fact 2.6 there exists $\lambda \in \mathcal{C}$ such that $qv = v\lambda$, for any $v \in \mathcal{V}$.

For $r \in \mathcal{R}, v \in \mathcal{V}$, we can write, $qv = v\lambda$, $r(qv) = r(v\lambda)$, and also $q(rv) = (rv)\lambda$. Thus $0 = [q, r]v$, for any $v \in \mathcal{V}$, i.e., $[q, r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible $\mathcal{R}$-module, we have $[q, r] = 0$, for all $r \in \mathcal{R}$, i.e., $q \in Z(\mathcal{R})$ and $\alpha = 0$, a contradiction. This completes the proof. □

Theorem 2.1. *Let $\mathcal{R}$ be a prime ring with center $Z(\mathcal{R})$, C be the extended centroid of $\mathcal{R}$ and $\mathcal{L}$ be a non-central Lie ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation δ associated with a nonzero derivation α such that $\delta(u)^m - (u)^n \in Z(\mathcal{R})$ for all $u \in \mathcal{L}$, where m, n are fixed positive integers, then $dim_{\mathcal{C}}\mathcal{R}\mathcal{C} = 4$.*

Proof. By [23, Theorem 3], there exists an element $a \in \mathcal{U}$ and a derivation α on $\mathcal{U}$ such that $\delta(x) = ax + \alpha(x)$ for all $x \in \mathcal{R}$. On the other hand, since $\mathcal{L}$ is non-central Lie ideal of $\mathcal{R}$, there exists a nonzero ideal $\mathcal{I}$ of $\mathcal{R}$ such that $[\mathcal{I}, \mathcal{I}] \subseteq \mathcal{L}$ unless $char(\mathcal{R}) = 2$ and $dim_{\mathcal{C}}\mathcal{R}\mathcal{C} = 4$ by Fact 2.4. Therefore, we may assume that $char(\mathcal{R}) \neq 2$ and $[\mathcal{I}, \mathcal{I}] \subseteq \mathcal{L}$ for a nonzero ideal $\mathcal{I}$ of $\mathcal{R}$. Since $\mathcal{I}$, $\mathcal{R}$ and $\mathcal{U}$ satisfy the same generalized differential identities, we have

$$[(a[x, y] + [\alpha(x), y] + [x, \alpha(y)])^m - ([x, y])^n, z] = 0\,.$$

for all $x, y, z \in \mathcal{U}$. We now divide the proof into two cases in view of Kharchenko's theorem (Fact 2.3).

Case 1. If α is an X-outer derivation, then

$$[(\mathrm{a}[x,y]+[s,y]+[x,t])^m-([x,y])^n,z]=0\,.$$

for all $x, y, z, s, t \in \mathcal{U}$. In particular, for $x = 0$, we get $[[s, y]^m, z] = 0$ for all $s, y, z \in \mathcal{U}$. Hence, we get the desired conclusion $dim_{\mathcal{C}}\mathcal{R}\mathcal{C} = 4$ by Fact 2.5.

Case 2. If α is an X-inner derivation, there exists a non-central $\mathrm{q} \in \mathcal{U}$ such that $\alpha(x) =$

$[\mathrm{q}, x]$ for all $x \in \mathcal{R}$. By hypothesis, we have

$$[((\mathrm{a}+\mathrm{q})[x,y]-[x,y]\mathrm{q})^m-[x,y]^n,z]=0\,.$$

for all $x, y, z \in \mathcal{U}$ as argued before. If, now, $\mathrm{a}+\mathrm{q} \in \mathcal{C}$, then

$$[([x,y]\mathrm{a})^m-[x,y]^n,z]=0\,.$$

for all $x, y, z \in \mathcal{U}$. Since $\mathrm{q} \notin \mathcal{C}$, we have $\mathrm{a} \notin \mathcal{C}$, and thus the last identity is a non-trivial generalized polynomial identity (GPI) for $\mathcal{U}$. If, on the other hand, $\mathrm{a}+\mathrm{q} \notin \mathcal{C}$, then the first identity above is a non-trivial GPI for $\mathcal{U}$. Therefore in any case $\mathcal{U}$ is a prime GPI-ring. We also note that, in the case when $\mathcal{C}$ is an infinite field, our initial identity is also satisfied by $\mathcal{U} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is the algebraic closure of $\mathcal{C}$. Since both $\mathcal{U}$ and $\mathcal{U} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ are centrally closed prime algebras ([12, Theorems 2.5 and 3.5]), we may replace $\mathcal{U}$ by either itself or $\mathcal{U} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ according as $\mathcal{C}$ is either finite or infinite. Therefore, we may assume that the center $\mathcal{C}$ of $\mathcal{U}$ is either finite or algebraically closed. Now by Martindale's theorem, $\mathcal{U}$ is a primitive ring with a nonzero socle $\mathcal{H}$. Hence $\mathcal{U}$, is a dense subring of the ring of all $\mathcal{C}$-endomorphisms of a vector space $\mathcal{V}$ over $\mathcal{C}$ (see e.g. [17]).

Suppose first that $dim_{\mathcal{C}}\mathcal{V}$ is infinite. Then

$$((\mathrm{a}+\mathrm{q})[x,y]-[x,y]\mathrm{q})^m-[x,y]^n \in \mathcal{C}\cap\mathcal{H}=(0)\,.$$

for all $x, y \in \mathcal{H}$. By Theorem 2.1 in [15], $\mathcal{H}$ is commutative, a contradiction. Therefore, $\mathcal{V}$ must be finite dimensional, say $dim_{\mathcal{C}}\mathcal{V} = \mathrm{k}$. Since $\mathcal{U}$ is non-commutative, we have $\mathcal{U} \cong \mathcal{M}_{\mathrm{k}}(\mathcal{C})$ and $\mathrm{k} \geq 2$. Finally, we need to show $\mathrm{k} = 2$. Suppose for the moment that $\mathrm{k} \geq 3$. Then

$$((\mathrm{a}+\mathrm{q})e_{\mathrm{ij}}-e_{\mathrm{ij}}q)^m-e_{\mathrm{ij}}^n \in \mathcal{C}.I_{\mathrm{k}}$$

for all distinct $1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{k}$. By commuting it with e_{ir} for any $\mathrm{r} \neq \mathrm{i}, \mathrm{j}$, one gets $\mathrm{q}_{\mathrm{ji}} = 0$, that is q is diagonal. Consider now the automorphism $\psi\colon x \mapsto (1 + e_{\mathrm{ij}})x(1 - e_{\mathrm{ij}})$ of $\mathcal{M}_{\mathrm{k}}(\mathcal{C})$ for $\mathrm{i} \neq \mathrm{j}$. Notice that

$$(\psi(\mathrm{a}+\mathrm{q})[x,y]-[x,y]\psi(q))^m-[x,y]^n \in \mathcal{C}.I_{\mathrm{k}}$$

for all $x, y \in \mathcal{M}_{\mathrm{k}}(\mathcal{C})$. From above argument $\psi(\mathrm{q})$ is a diagonal matrix. Set $\mathrm{q} = \sum_{i=1}^{\mathrm{k}}\lambda_{\mathrm{i}}e_{\mathrm{ii}}$. Since $\psi(\mathrm{q}) - \mathrm{q} = (\lambda_{\mathrm{j}} - \lambda_{\mathrm{i}})e_{ij}$ is a diagonal matrix, we must have $\lambda_{\mathrm{i}} = \lambda_{\mathrm{j}}$ for all $\mathrm{i} \neq \mathrm{j}$,

which contradicts to choice of q. Thus, $\mathcal{U}$ as well as R satisfy the standard polynomial identity s_4. Hence as it is well-known, $dim_{\mathcal{C}}\mathcal{R}\mathcal{C} = 4$ since $\mathcal{R}$ is non-commutative. This completes the proof. □

Theorem 2.2. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2 and $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation δ associated with a nonzero derivation α such that $[\delta(x), \alpha(y)]_m = \delta([x, y])^n$, for all $x, y \in \mathcal{I}$, where m, n are fixed positive integers, then $\mathcal{R}$ is commutative.*

Proof. By the given hypothesis and using [23, Theorem 3], we can write

$$[ax + \alpha(x), \alpha(y)]_m = (a[x, y] + \alpha([x, y]))^n \text{ for all } x, y \in \mathcal{I} ,$$

i.e.,

$$[ax, \alpha(y)]_m + [\alpha(x), \alpha(y)]_m = (a[x, y] + [\alpha(x), y] + [x, \alpha(y)])^n .$$

In view of Kharchenko's theorem (Fact 2.3), we split the proof into two cases:
If α is not $\mathcal{Q}$-inner, then $\mathcal{I}$ satisfies the polynomial identity

$$[ax, t]_m + [s, t]_m = (a[x, y] + [s, y] + [x, t])^n, \text{ for all } x, y, s, t \in \mathcal{I} .$$

In particular, for $x = y = 0$, we see that $\mathcal{I}$ satisfies the blended component $[s, t]_m = 0$, which is rewritten as $[I_x(y), y]_{k-1} = 0$. By Lanski [20, Theorem 1], either R is commutative, or $I_x = 0$ i.e., $I \subseteq Z(R)$ in which R is also commutative by Mayne [28, Lemma 3].

Next, assume that α is $\mathcal{Q}$-inner induced by an element $q \in \mathcal{Q}$ i.e., $\alpha(x) = [q, x]$, for all $x \in \mathcal{R}$. Therefore, we have

$$[ax, [q, y]]_m + [[q, x], [q, y]]_m = (a[x, y] + [q, [x, y]])^n, \text{ for all } x, y \in \mathcal{I} .$$

By Chuang [6, Theorem 1], $\mathcal{I}$ and $\mathcal{Q}$ satisfy same generalized polynomial identities (GPIs) i.e.,

$$[ax, [q, y]]_m + [[q, x], [q, y]]_m = (a[x, y] + [q, [x, y]])^n, \text{ for all } x, y \in \mathcal{Q} .$$

When the center $\mathcal{C}$ of $\mathcal{Q}$ is infinite, we have

$$[ax, [q, y]]_m + [[q, x], [q, y]]_m = (a[x, y] + [q, [x, y]])^n ,$$

for all $x, y \in \mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is algebraic closure of $\mathcal{C}$. Since both $\mathcal{Q}$ and $\mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ are prime and centrally closed [12, Theorem 2.5 and Theorem 3.5], we may replace $\mathcal{R}$ by $\mathcal{Q}$ or $\mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ according as $\mathcal{C}$ is finite or infinite. Thus, we may assume that $\mathcal{R}$ is centrally closed over $\mathcal{C}$ ($i.e.$, $\mathcal{R}\mathcal{C} = \mathcal{R}$) which is either finite or algebraically closed and

$$[ax, [q, y]]_m + [[q, x], [q, y]]_m = (a[x, y] + [q, [x, y]])^n, \text{ for all } x, y \in \mathcal{R} .$$

Since $q \notin \mathcal{C}$, we can easily see that the last identity is a non-trivial generalized polynomial identity (GPI) for $\mathcal{R}$. By Martindale [25, Theorem 3], $\mathcal{RC}$ (and so $\mathcal{R}$) is a primitive ring having nonzero socle $\mathcal{H}$ with $\mathcal{C}$ as the associated division ring. Hence, by Jacobson's theorem [17, p.75], $\mathcal{R}$ is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{C}$ and $\mathcal{H}$ consists of the finite rank linear transformations in $\mathcal{R}$. If $\mathcal{V}$ is finite dimensional over $\mathcal{C}$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong \mathcal{M}_k(\mathcal{C})$, where $k = dim_{\mathcal{C}}\mathcal{V}$.

Suppose that $dim_{\mathcal{C}}\mathcal{V} \geq 2$, otherwise we have done. First, we want to show that v and qv are linearly $\mathcal{C}$-dependent, for all $v \in \mathcal{V}$. If $qv = 0$, then the set $\{v, qv\}$ is linearly $\mathcal{C}$-dependent. Suppose on contrary that v and qv are linearly $\mathcal{C}$-independent, for some $v \in \mathcal{V}$.

If $q^2v \notin span_{\mathcal{C}}\{v, qv\}$, then the set $\{v, qv, q^2v\}$ is linearly $\mathcal{C}$-independent. By the density of $\mathcal{R}$, there exists $x, y \in \mathcal{R}$ such that

$$\begin{array}{lll} xv = 0, & xqv = qv, & xq^2v = 0\,; \\ yv = 0, & yqv = v, & yq^2v = 2qv\,. \end{array}$$

Therefore,

$$\begin{aligned} 0 &= ([ax, [q, y]]_m + [[q, x], [q, y]]_m)\,v = (a[x, y] + [q, [x, y]])^n\,v \\ &= v,\ \text{a contradiction}\,. \end{aligned}$$

If $q^2v \in span_{\mathcal{C}}\{v, qv\}$, then $q^2v = \beta v + \gamma qv$, for some $\beta, \gamma \in \mathcal{C}$. Since v and qv are linearly $\mathcal{C}$-independent, by the density of $\mathcal{R}$, there exist $x, y \in \mathcal{R}$ such that

$$xv = 0,\ xqv = qv;\ yv = 0,\ yqv = v\,.$$

Thus,

$$\begin{aligned} (-1)^{m+1}(2^m qv - 2^{m-1}\gamma v) &= ([ax, [q, y]]_m)v + ([[q, x], [q, y]]_m)v \\ &= (a[x, y] + [q, [x, y]])^n v = 0\,, \end{aligned}$$

for some $\gamma \in \mathcal{C}$. This leads to the contradiction $(-2)^m qv + (1 + (-2)^{m-1}\gamma)v = 0$. So for each $v \in \mathcal{V}$, $qv = v\mu$, for some $\mu \in \mathcal{C}$. By a standard argument, it is easy to see that μ is independent of the choice of $v \in \mathcal{V}$. Thus, we can write $qv = v\mu$, for all $v \in \mathcal{V}$ and fixed $\mu \in \mathcal{C}$. Using the same arguments as in the proof of the Lemma 2.1, we conclude that $\alpha = 0$, a contradiction. This completes the proof. ☐

We immediately get the following corollary from the above theorem:

Corollary 2.1. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2 and m, n be the fixed positive integers. If $\mathcal{R}$ admits a generalized derivation δ associated with a nonzero derivation α such that $[\delta(x), \alpha(y)]_m = (\delta([x, y]))^n$, for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

3 The results in Semiprime Rings

From now on, $\mathcal{R}$ is a semiprime ring, $\mathcal{U}$ is the left Utumi quotient ring of $\mathcal{R}$. For developing the proof of the main theorem, we require the following facts.

Fact 3.1 ([2, Proposition 2.5.1]). *Any derivation of a semiprime ring $\mathcal{R}$ can be uniquely extended to a derivation of its left Utumi quotient ring $\mathcal{U}$, and so any derivation of $\mathcal{R}$ can be defined on the whole $\mathcal{U}$.*

Fact 3.2 ([7, p.38]). *If $\mathcal{R}$ is a semiprime ring, then its left Utumi quotient ring is also semiprime. The extended centroid $\mathcal{C}$ of a semiprime ring coincides with the center of its left Utumi quotient ring.*

Fact 3.3 ([7, p.42]). *Let $\mathcal{B}$ be the set of all the idempotents in $\mathcal{C}$, the extended centroid of $\mathcal{R}$. Suppose that $\mathcal{R}$ is orthogonally complete $\mathcal{B}$-algebra. Then for any maximal ideal $\mathcal{P}$ of $\mathcal{B}$, $\mathcal{P}\mathcal{R}$ forms a minimal prime ideal of $\mathcal{R}$, which is invariant under any derivation of $\mathcal{R}$.*

Fact 3.4 ([22]). *If $\mathcal{I}$ is a two-sided ideal of $\mathcal{R}$, then $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{U}$ satisfy the same generalized polynomial identities.*

For a complete and detailed description of the theory of generalized polynomial identities involving derivations, we refer the reader to [2, Chapter 7]. Now we prove the following result.

Theorem 3.1. *Let $\mathcal{R}$ be a semiprime ring of characteristic different from 2 with center $Z(\mathcal{R})$. If $\mathcal{R}$ admits a generalized derivation δ associated with a nonzero derivation α such that $[\delta(x), \alpha(y)]_m = \delta([x, y])^n$, for all $x, y \in \mathcal{R}$, where m, n are fixed positive integers, then there exists a central idempotent element e in $\mathcal{U}$ such that on the direct sum decomposition $\mathcal{U} = \mathrm{e}\mathcal{U} \oplus (1-\mathrm{e})\mathcal{U}$, α vanishes identically on $\mathrm{e}\mathcal{U}$ and the ring $(1-\mathrm{e})\mathcal{U}$ is commutative.*

Proof. Since $\mathcal{R}$ is semiprime and δ is a generalized derivation of $\mathcal{R}$, by Lee [23, Theorem 3], there exists an element $\mathrm{a} \in \mathcal{U}$ and a derivation α on $\mathcal{U}$ such that $\delta(x) = \mathrm{a}x + \alpha(x)$ for all $x \in \mathcal{R}$. Therefore, we have $[\mathrm{a}x + \alpha(x), \alpha(y)]_m = (\mathrm{a}[x, y] + \alpha([x, y]))^n$, for all $x, y \in \mathcal{R}$. By Fact 3.2, $Z(\mathcal{U}) = \mathcal{C}$, the extended centroid of $\mathcal{R}$, and by Fact 3.1, the derivation α can be uniquely extended on $\mathcal{U}$. By Lee [22, Theorem 3], $\mathcal{R}$ and $\mathcal{U}$ satisfy the same differential identities. Thus, $[\mathrm{a}x+\alpha(x), \alpha(y)]_m-(\mathrm{a}[x, y]+\alpha([x, y]))^n = 0$, for all $x, y \in \mathcal{U}$. Let $\mathcal{B}$ be the complete Boolean algebra of idempotents in $\mathcal{C}$ and $\mathcal{M}$ be any maximal ideal of $\mathcal{B}$. Therefore, by Chuang [7, p.42], $\mathcal{U}$ is orthogonally complete $\mathcal{B}$-algebra, and by Fact 3.3, $\mathcal{M}\mathcal{U}$ is a prime ideal of $\mathcal{U}$, which is α-invariant. Let $\overline{\alpha}$ be the derivation induced by α on $\overline{\mathcal{U}} = \mathcal{U}/\mathcal{M}\mathcal{U}$, i.e., $\overline{\alpha}(\overline{u}) = \overline{\alpha(u)}$, for all $u \in U$. Then for all $\overline{x}, \overline{y} \in \overline{\mathcal{U}}$, $[(\overline{\mathrm{a}}\overline{x} + \overline{\alpha}(\overline{x}), \overline{\alpha}(\overline{y})]_m - (\overline{\mathrm{a}}[\overline{x}, \overline{y}] + \overline{\alpha}([\overline{x}, \overline{y}]))^n = 0$. It is obvious that $\overline{\mathcal{U}}$ is prime. Therefore, by Corollary 2.1, we have either $\overline{\mathcal{U}}$ is commutative or $\overline{\alpha} = 0$, i.e., either $\alpha(\mathcal{U}) \subseteq \mathcal{M}\mathcal{U}$ or

$[\mathcal{U}, \mathcal{U}] \subset \mathcal{M}\mathcal{U}$. Hence, $\alpha(\mathcal{U})[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{M}\mathcal{U}$, where $\mathcal{M}\mathcal{U}$ runs over all minimal prime ideals of $\mathcal{U}$. Since $\bigcap_{\mathcal{M}} \mathcal{M}\mathcal{U} = 0$, we obtain $\alpha(\mathcal{U})[\mathcal{U}, \mathcal{U}] = 0$.

By using the theory of orthogonal completion for semiprime rings [2, Chapter 3], it is clear that there exists a central idempotent element e in $\mathcal{U}$ such that on the direct sum decomposition $\mathcal{U} = \mathrm{e}\mathcal{U} \oplus (1-\mathrm{e})\mathcal{U}$, α vanishes identically on $\mathrm{e}\mathcal{U}$ and the ring $(1-\mathrm{e})\mathcal{U}$ is commutative. This completes the proof. □

4 Applications on Banach algebras

This section deals with the applications of our main results. Here, $\mathfrak{A}$ will be denoted as a complex Banach algebra and δ is a generalized derivation on $\mathfrak{A}$. Let us state some well known and elementary definitions for the sake of completeness.

By Banach algebra, we shall mean that complex normed algebra $\mathfrak{A}$ whose underlying vector space is a Banach space. The Jacobson radical $rad(\mathfrak{A})$ of $\mathfrak{A}$ is the intersection of all primitive ideals. If the Jacobson radical reduces to the zero element, then $\mathfrak{A}$ is called semisimple. In fact, any Banach algebra $\mathfrak{A}$ without a unity can be embedded into a unital Banach algebra $\mathfrak{A}_I = \mathfrak{A} \oplus \mathbb{C}$ as an ideal of codimension one. In particular, we may identify $\mathfrak{A}$ with the ideal $\{(x, 0) : x \in \mathfrak{A}\}$ in $\mathfrak{A}_I$ via the isometric isomorphism $x \mapsto (x, 0)$.

In this section, we apply the purely algebraic results which is derived in section 2 and obtain the condition that every continuous derivation on a Banach algebra maps into the radical. The proofs of the results rely on a Sinclair's theorem [35] which stated that every continuous derivation α of a Banach algebra $\mathfrak{A}$ leaves the primitive ideals of $\mathfrak{A}$ invariant. As we have mentioned before, Thomas [37] generalized the Singer-Wermer theorem by proving that any derivation on a commutative Banach algebra maps the algebra into its radical. This result leads us an important question whether the theorem can be proved without using any commutativity assumption. On this note, there are many papers [25, 26, 35] which shows that the theorem holds without any commutativity assumption. We also acquire that every derivation maps into its radical with some property without any commutativity assumption.

Our main result in this section concerns about the continuous generalized derivations on Banach algebra.

Theorem 4.1. *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and m, n be the fixed positive integers. Suppose $\delta = L_{\mathrm{a}} + \alpha$ is a continuous generalized derivation of $\mathfrak{A}$ for some element $\mathrm{a} \in \mathfrak{A}$ and α is a derivation on $\mathfrak{A}$. If $[\delta(x), \alpha(y)]_m - \delta([x, y])^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, then $\alpha(\mathfrak{A}) \subseteq rad(\mathfrak{A})$.*

Proof. Since δ is continuous and it is well known that the left multiplication map is continuous, we find that the derivation α is continuous. In [35], Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant.

Thus, for any primitive ideal $\mathcal{P}$ of $\mathfrak{A}$, it follows that $\delta(\mathcal{P}) \subseteq \mathrm{a}\mathcal{P} + \alpha(\mathcal{P}) \subseteq \mathcal{P}$. It means that the continuous generalized derivation δ leaves the primitive ideals invariant. Hence, we can introduce a generalized derivation $\delta_{\mathcal{P}} : \overline{\mathfrak{A}} \to \overline{\mathfrak{A}}$ by $\delta_{\mathcal{P}}(\overline{x}) = \delta_{\mathcal{P}}(x + \mathcal{P}) = \delta_{\mathcal{P}}(x) + \mathcal{P} = \mathrm{a}x + \alpha(x) + \mathcal{P}$, for all $x \in \mathfrak{A}$ and $\overline{x} = x + \mathcal{P}$, where $\mathfrak{A}/\mathcal{P} = \overline{\mathfrak{A}}$ is a factor Banach algebra, for any primitive ideals $\mathcal{P}$. Moreover, as $[\mathrm{a}x + \alpha(x), \alpha(y)]_m - (\mathrm{a}[x, y] + \alpha([x, y]))^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, it follows that $[(\overline{\mathrm{a}x} + \overline{\alpha}(\overline{x}), \overline{\alpha}(\overline{y})]_m - (\overline{\mathrm{a}}[\overline{x}, \overline{y}] + \overline{\alpha}([\overline{x}, \overline{y}]))^n = \overline{0}$, for all $\overline{x}, \overline{y} \in \overline{\mathfrak{A}}$. Due to primitiveness of $\overline{\mathfrak{A}}$ it is prime. Thus, by Corollary 2.1, either $\overline{\mathfrak{A}}$ is commutative or $\overline{\alpha} = \overline{0}$ i.e., $[\mathfrak{A}, \mathfrak{A}] \subseteq \mathcal{P}$ or $\alpha(\mathfrak{A}) \subseteq \mathcal{P}$.

Now, let $\mathcal{P}$ be a primitive ideal such that $\overline{\mathfrak{A}}$ is commutative. Singer and Wermer [36] proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Moreover, by a result of Johnson and Sinclair [18], any linear derivation on a semisimple Banach algebra is continuous. Hence, there is no nonzero linear continuous derivation on commutative semisimple Banach algebras. Therefore, $\overline{\alpha} = \overline{0}$ in $\overline{\mathfrak{A}}$ and hence, in any case, we get $\alpha(\mathfrak{A}) \subseteq \mathcal{P}$, for all primitive ideal $\mathcal{P}$ of $\mathfrak{A}$. Since radical $rad(\mathfrak{A})$ of $\mathfrak{A}$ is the intersection of all primitive ideals, we get the required conclusion. □

Using arguments similar to those used in the proof of the Theorem 1.4, we may conclude with the following (we omit the details of the proof). We can prove.

Corollary 4.1. *Let $\mathfrak{A}$ be a non-commutative Banach algebra and m, n be the fixed positive integers. Suppose $\delta = L_{\mathrm{a}} + \alpha$ is a continuous generalized derivation of $\mathfrak{A}$, where $\mathrm{a} \in \mathfrak{A}$ and α is a derivation of $\mathfrak{A}$. If $[\delta(x), \alpha(y)]_m = (\delta([x, y]))^n$, for all $x, y \in \mathfrak{A}$, then $\alpha(\mathfrak{A}) \subseteq rad(\mathfrak{A})$, the Jacobson radical of $\mathfrak{A}$. In particular, if $\mathfrak{A}$ is semisimple, then $\alpha = 0$.*

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Proper Categories and Their Duals

Abstract: Let $\mathcal{C}$ be a category with subobjects in which every inclusion splits and every morphism has factorization. A cone γ in category $\mathcal{C}$ is certain map from $v\mathcal{C}$ to $\mathcal{C}$ and a cone γ in $\mathcal{C}$ is a proper cone if there is at least one component of γ an epimorphism. Here we introduce the semigroup $\mathcal{P}C$ of proper cones in a category $\mathcal{C}$ and its dual $\mathcal{P}^*C$. Further it is also shown that the categories of left [right] ideals $\mathbb{L}(S)$ [$\mathbb{R}(S)$] of a semigroup S with appropriate morphisms are proper categories. The semigroup $\mathcal{P}\mathbb{L}(S)$ [$\mathcal{P}\mathbb{R}(S)$] of proper cones and their duals in these categories are also discussed.

Keywords: Proper category; Cones; Ideals categories of semigroup.

1 Introduction

In [2] Nambooripad introduced a normal category as a category $\mathcal{C}$ with subobjects in which every inclusion splits, each morphism admits a factorization of the form euj where e is a retraction, u an isomorphism and j an inclusion. A norml cone γ with vertex $d \in \mathcal{C}$ is a map from $v\mathcal{C}$ to $\mathcal{C}$ such that γ is a cone and there exists at least one $c \in v\mathcal{C}$ such that $\gamma(c)\colon\ c \to d$ is an isomorphism. It is shown that the semigroup of normal cones $\mathcal{T}C$ in a normal category $\mathcal{C}$ is a regular semigroup. Further in [2] it is shown that the category $\mathbb{L}(S)$ [$\mathbb{R}(S)$] whose objects are principal left [right] ideals generated by idempotents of a regular semigroup S and morphisms are appropriate right [left] translations are normal categories. Making use of normal categories their duals and some nice relations called cross-connections that exsisted between such categories Nambooripad described the structure of regular semigroups. Extending this theory to include some class of non regular semigroups, in [3] P.G.Romeo introduced a balanced category $\mathcal{C}$ as a category with subobjects in which every inclusion splits, each morphism admits a balanced factorization. That is, factorization of the form euj where e is a retrction, u is a balanced morphism (ie., a morphism which is both monic and epi) and j an inclusion. A balanced cone γ with vertex $d \in \mathcal{C}$ is a cone in $\mathcal{C}$ such that there exists at least one $c \in v\mathcal{C}$ such that $\gamma(c)\colon\ c \to d$ is a balanced morphism. It is shown that the category $\mathbb{L}(S)$ [$\mathbb{R}(S)$] whose objects are principal left [right] ideals generated by idempotents of a concordant semigroup S and morphisms are appropriate right [left] translations are balanced categories.

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In this paper we further generalise these ideas and define a proper category to study the structure theory of semigroups. A proper category $\mathcal{C}$ is a category with sub objects, every inclusion splits and every morphism has factorization qj where q is epi and j an inclusion. A cone γ in such a category is said to be proper there exists at least one $c \in v\mathcal{C}$ such that $\gamma(c)\colon c \to d$ is an epimorphism. The set of all proper cones in a proper category is the semigroup $\mathcal{PC}$ and its dual $\mathcal{P}^*\mathcal{C}$. The categories $\mathbb{L}(\mathcal{PC})$ and $\mathbb{R}(\mathcal{PC})$ of left and right ideals of the semigroups $\mathcal{PC}$ are shown to be proper categories and their dual categories are also described.

2 Categories, Functors and Functor Categories

In the following we recall some definitions and results regarding categories, however for any notations or results not explicitly stated the reader is referred to S. Maclane (cf.[1]).

A category $\mathcal{C}$ consists of object $\{A, B, \dots\}$ and morphisms $\{f, g, \dots\}$ such that for each arrow f there are objects $dom(f)$, $cod(f)$ called the domain and codomain of f. Given arrows f and g with $cod(f) = dom(g)$ then there exists $f \cdot g$ called the composite of f and g. For each object A the arrow $I_A\colon A \to A$ called the identity arrow. The category whose objects are same as that of $\mathcal{C}$ and morphisms are f^{op} with domain the $cod(f)$ and codomain of f^{op} is the $dom(f)$ where f is any morphism in $\mathcal{C}$ is called the opposite category of $\mathcal{C}$ and is written as $\mathcal{C}^{op}$

Set, Grp are familiar examples of categories whose objects and morphisms are sets and set maps, groups and group homomorphisms respectively. Let A, B be objects in a category $\mathcal{C}$ then the set consists of all morphisms with domain A and codomain B is called the *hom-set* and is denoted as $hom_{\mathcal{C}}(A; B)$ or $hom(A; B)$.

A (covariant)functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping of objects to objects and arrows to arrows such that if $f\colon c \to c' \in \mathcal{C}$ then $F(f)\colon F(c) \to F(c')$, $F(I_c) = I_{F(c)}$ and $F(g \circ f) = F(g) \circ F(f)$. By a functor we always means a covariant functor. A (covariant) functor $F\colon \mathcal{C}^{op} \to \mathcal{D}$ is called a contravariant functor of $\mathcal{C}$ to $\mathcal{D}$.

Example 1. *Consider $P\colon$ **Set** $\to$ **Set** which assigns to each set X the usual power set $P(X)$, of all subsets $S \subset X$; its arrow function assigns to each $f\colon X \to Y$ a map $P(f)\colon P(X) \to P(Y)$ which sends each $S \subset X$ to its image $fS \subset Y$. Since $P(I_X) = I_{P(X)}$ and $P(g \circ f) = P(g) \circ P(f)$, this defines a functor $P\colon$ **Set** $\to$ **Set** known as the power set functor.*

Example 2. *Each object $A \in \mathcal{C}$ defines a functor $hom(A; -)\colon \mathcal{C} \to$ **Set** by*

$$hom(A; B) = \{f\colon A \to B\} \in \textbf{Set} \text{ for } B \in \mathcal{C}$$

is called the representable functor of A (or the covariant hom-functor determined by A). Clearly

$$hom(A; I_X) = I_{hom(A;X)} \text{ and}$$
$$hom(A; g \circ f) = hom(A; g) \circ hom(A; f)\ .$$

A functor $T\colon \mathcal{C} \to \mathcal{D}$ is an isomorphism if and only if there is a functor $S\colon \mathcal{D} \to \mathcal{C}$ for which both composites $S \circ T$ and $T \circ S$ are identity functors.

Definition 2.1. *Given two functors $S, T\colon \mathcal{C} \to \mathcal{B}$, a natural transformation $\tau\colon S \to T$ is a function which assigns to each object c of $\mathcal{C}$ an arrow $\tau_c\colon Sc \to Tc$ of $\mathcal{B}$ in such a way that every $f\colon c \to c'$ in $\mathcal{C}$ yields $Tf \circ \tau_c = \tau_{c'} \circ Sf$.*

A natural transformation is often called a morphism of functors. $\mathcal{C}$ and $\mathcal{D}$ be two categories then $[\mathcal{C}; \mathcal{D}]$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms are natural transformations. If S and T are functors, the hom-set of this category is

$$Nat(S; T) = \{\eta \mid \eta\colon S \to T \text{ is natural}\}$$

and the composition is the component-wise product of natural transformations. Any subcategory of $[\mathcal{C}; \mathcal{D}]$ will be called a *functor category* from $\mathcal{C}$ to $\mathcal{D}$ and $\mathcal{C}^*$ denote the functor category $[\mathcal{C}; \textbf{Set}]$.

Duality is a correspondence between properties of category $\mathcal{C}$ and dual properties of the opposite category $\mathcal{C}^{op}$. Given a statement regarding the category $\mathcal{C}$, by interchanging the source and target of each morphism as well as changing the order of composition the dual statement is obtained regarding the opposite category $\mathcal{C}^{op}$. If a statement is true about $\mathcal{C}$, then its dual statement is true about $\mathcal{C}^{op}$.

Definition 2.2. *A morphism f in a category $\mathcal{C}$ is a monomorphism if for $g, h \in \mathcal{C}$, $gf = hf$ implies $g = h$; that is f is a monomorphism if it is right cancelable.*

A morphism $f \in \mathcal{C}$ is an epimorphism if f is left cancelable and f is a balanced morphism if it is both mono and epi.

2.1 Category with Subobjects

A pre order P is a small category such that for all $p, q \in vP$, $P(p, q)$ contains atmost one morphism. $p \preceq q \Leftrightarrow P(p, q) \neq \phi$. Clearly $p \preceq q$ is a quasi order on the vP. The pre order P is said to be strict if the quasi order $\preceq$ is a partial order. A choice of sub objects in a category $\mathcal{C}$ is a sub pre order $P \subseteq \mathcal{C}$ satisfying

1. P is a strict pre order with $vP = v\mathcal{C}$,
2. every $f \in P$ is a monomorphism in $\mathcal{C}$,
3. if $f, g \in P$ and if $f = hg$ for some $h \in \mathcal{C}$ then $h \in P$.

If P is a choice of sub objects in $\mathcal{C}$, the pair $(\mathcal{C}, P)$ is called a category with sub objects. Here after we regard $\mathcal{C}$ as a category with sub objects. For each $c \in \mathcal{C}$, denote by $\langle c \rangle_{\mathcal{C}}$ the full subcategory of $\mathcal{C}$ whose objects are sub objects of $c \in \mathcal{C}$.

In the case of "algebraic categories" like **Grp**, $\mathbf{Vct_K}$, $\mathbf{Mod_R}$, the sub objects can be chosen in such a way that the set of all monomorphisms whose underlying maps

are inclusions and in this case all monomorphisms are embeddings. In the case of "topological categories" like **Top**, **Tvs**, **Lcs** choice of sub objects by this method include monomorphisms which are not embeddings.

A category $\mathcal{C}$ is said to have factorization property if every $f \in \mathcal{C}$ can be expressed as $f = pm$ where p is an epimorphism and m is an embedding. A factorization of the form $f = qj$ where q is an epimorphism and j is an inclusion is called a *canonical factorization*. A category $\mathcal{C}$ is said to have factorization property if and only if every morphism in $\mathcal{C}$ has factorization.

The categories **Set**, **Grp**, $\mathbf{Vct_K}$, **Top**, **Tvs** have canonical factorizations.

Proposition 2.1. *Let $\mathcal{C}$ be a category with factorization property. Then*
1. *$f \in \mathcal{C}$ has image if and only if there exists a unique canonical factorization $f = xj$.*
2. *if every inclusion splits then every morphism in $\mathcal{C}$ has a unique canonical factorization.*

Definition 2.3. *Let $\mathcal{C}$ be a category and $d \in \mathrm{v}\mathcal{C}$. A map $\gamma\colon \mathrm{v}\mathcal{C} \to \mathcal{C}$ is a cone from the base $\mathrm{v}\mathcal{C}$ to the vertex d if γ satisfies the following:*
1. *$\gamma(c) \in \mathcal{C}(c, d)$ for all $c \in \mathrm{v}\mathcal{C}$.*
2. *If $c' \subseteq c$ then $j^c_{c'}\gamma(c) = \gamma(c')$.*

Given the cone γ we denote by c_γ the vertex of γ and for each $c \in v\mathcal{C}$, the morphism $\gamma(c)\colon c \to c_\gamma$ is called the component of γ at c.

Proposition 2.2. *Let γ be a cone in $\mathcal{C}$ and $f \in \mathcal{C}(c_\gamma, c)$. Then $\gamma \star f^o$ is the cone*

$$\gamma \star f^o(c') = \gamma(c')f^o \quad \textit{for all} \quad c' \in v\mathcal{C}$$

with vertex Imf such that for all composable pair of morphisms $f, g \in \mathcal{C}$ with $domf = c_\gamma$ then

$$\gamma \star (fg)^o = (\gamma \star f^o) \star (g|Imf)^o \ .$$

Proof. Clearly $\gamma \star f^o$ is a cone. Let $\eta = \gamma \star f^o$ so that $c_\eta = Imf = c'$(say). Then

$$\begin{aligned}(\gamma \star (fg)^o)(a) &= \gamma(a)(f^o(g|c')^o)\\ &= (\gamma(a)f^o)(g|c')^o\\ &= (\eta \star g|c')^o(a)\end{aligned}$$

for all $a \in v\mathcal{C}$. ☐

We denote the set of all cones in the category $\mathcal{C}$ by $C\mathcal{C}$. For $\gamma \in C\mathcal{C}$, let

$$N_\gamma = \{c \in v\mathcal{C}\colon \gamma(c) \text{ is epimorphism }\}$$

$$B_\gamma = \{c \in v\mathcal{C}\colon \gamma(c) \text{ is a balanced morphism}\} \ .$$

$$M_\gamma = \{c \in v\mathcal{C}\colon \gamma(c) \text{ is an isomorphism}\} \ .$$

Clearly $M_\gamma \subseteq B_\gamma \subseteq N_\gamma$ for all $\gamma \in C\mathcal{C}$.

3 Proper Categories

Let $\mathcal{C}$ be a category with sub objects in which inclusions splits and every morphism has canonical factorization. Then the proper cones in $\mathcal{C}$ written as $\mathcal{PC}$ is the set of all cones $\gamma \in \mathcal{C}$ such that there exists at least one $c \in v\mathcal{C}$ with $\gamma(c): c \to c_\gamma$ is epi (that is., $\gamma(c) = \gamma(c)^o$). In particular if $\gamma(c): c \to c_\gamma$ is balanced morphism [isomorpism] then γ is balanced [normal] cone respectively.

Definition 3.1. *A small category $\mathcal{C}$ with sub objects is called proper category if it satisfies the following:*
1. *every inclusion in $\mathcal{C}$ splits,*
2. *every morphism $f \in \mathcal{C}$ has unique canonical factorization and*
3. *for each $a \in v\mathcal{C}$, there exists $\gamma \in \mathcal{PC}$ such that $\gamma(a) = I_a$.*

Proposition 3.1. *The set of all proper cones $\mathcal{PC}$ in the category $\mathcal{C}$ is a semigroup with respect to the binary operation defined*

$$\gamma \cdot \eta = \gamma \star \eta(c_\gamma)^o$$

where $\gamma, \eta \in P\mathcal{C}$.

Proof. Clearly $\gamma \star \eta(c_\gamma)^o$ is a cone. To prove that it is proper, let $c \in N_\gamma$ so $\gamma(c)$ is an epimorphism and hence $\gamma(c)\eta(c_\gamma)^o$ is an epimorphism, thus $c \in N_{\gamma\star\eta(c_\gamma)^o}$. □

Proposition 3.2. *$\gamma \in \mathcal{PC}$ is an idempotent proper cone if and only if $\gamma(c_\gamma) = I_{c_\gamma}$*

Proof. Suppose γ is an idempotent proper cone and let $c \in N_\gamma$, then

$$\gamma(c)(\gamma(c_\gamma))^o = (\gamma \cdot \gamma)(c) = \gamma(c) .$$

Since $\gamma(c)$ is an epimorphism $(\gamma(c_\gamma))^o = I_{c_\gamma}$, $\gamma(c_\gamma) \in \mathcal{C}(c_\gamma, c_\gamma)$ and so $\gamma(c_\gamma) = I_{c_\gamma}$.

Conversely, if $\gamma(c_\gamma) = I_{c_\gamma}$, then for every $a \in v\mathcal{C}$

$$(\gamma \cdot \gamma)(a) = \gamma(a)(\gamma(c_\gamma))^o = \gamma(a)I_{c_\gamma} = \gamma(a)$$

hence γ is idempotent. □

Note that

$$\mathcal{TC} \subseteq \mathcal{BC} \subseteq \mathcal{PC}$$

where $\mathcal{BC}$, $\mathcal{TC}$ denotes the semigroups of balanced cones and normal cones respectively A cone γ in $C\mathcal{C}$ is proper, balanced or normal according as $N_\gamma \neq \phi$, $B_\gamma \neq \phi$ or $M_\gamma \neq \phi$ respectively.

3.1 Green's Relations on the Semigroup of Proper Cones

Let S be a semigroup, for $a, b \in S$, $a\mathcal{L}b$ if and only if a and b generate the same principal left ideals and $a\mathcal{R}b$ if and only if they generate the same principal right ideals.

The relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. The set $S/\mathcal{L}$ of all $\mathcal{L}$- classes of S is order isomorphic with the partially ordered set of all principal left ideals of the semigroup S under inclusion, ie., for $L_x, L_y \in S/\mathcal{L}$.

$$L_x \leq L_y \Leftrightarrow S^1 x \subseteq S^1 y \,.$$

Dually the inclusion among the principal right ideals induces a partial order on the set $S/\mathcal{R}$.

Lemma 3.1. *Let $\gamma \in \mathcal{PC}$ and $\epsilon \in E(\mathcal{PC})$. Then $\epsilon \cdot \gamma = \gamma$ if and only if there exists a unique epimorphism $f\colon c_\epsilon \to c_\gamma$ such that $\gamma = \epsilon \star f$ and $c_\epsilon \in N_\gamma$.*

Proof. Let $\epsilon \cdot \gamma = \gamma$. Then for $c \in v\mathcal{C}$,

$$\gamma(c) = (\epsilon \cdot \gamma)(c) = \epsilon(c)(\gamma(c_\epsilon))^o$$

taking $c = c_\epsilon$, $\gamma(c_\epsilon) = (\gamma(c_\epsilon))^o$ which implies $f = \gamma(c_\epsilon)$ is an epimorphism and $\gamma = \epsilon \star f$. Conversely assume that $\gamma = \epsilon \star f$ where $f \in \mathcal{C}(c_\epsilon, c_\gamma)$ is an epimorphism, then $\gamma(c_\epsilon) = f$ and for all $c \in v\mathcal{C}$

$$(\epsilon \cdot \gamma)(c) = \epsilon(c) f = \gamma(c)$$

If $\epsilon \star f = \epsilon \star g$ for epimorphisms f and g then $f = \gamma(c_\epsilon) = g$. □

The following proposition shows that the partially ordered set of left ideals of $\mathcal{PC}$ is order isomorphic with $v\mathcal{C}$ when the category $\mathcal{C}$ is proper.

Proposition 3.3. *Let $\mathcal{C}$ be a proper category. If $\gamma, \gamma' \in \mathcal{PC}$ and if $L_\gamma \leq L_{\gamma'}$ then $c_\gamma \subseteq c_{\gamma'}$. The converse holds if γ, γ' are normal cones.*

Proof. If $L_\gamma \leq L_{\gamma'}$ then $\gamma = \tau \cdot \gamma'$ for some $\tau \in \mathcal{PC}$ and so $\gamma = \tau \star f^o$ for some $f \in \mathcal{C}(c_\tau, c_{\gamma'})$. This implies that $c_\gamma = imf \subseteq c_{\gamma'}$. □

Definition 3.2. *Let $\mathcal{C}$ be a proper category. For each $\gamma \in \mathcal{PC}$, define $H(\gamma; -)\colon \mathcal{C} \to Set$ by*

$$H(\gamma; c) = \{\gamma \star f^o : f \in \mathcal{C}(c_\gamma, c)\}$$
$$H(\gamma; g)\colon \gamma \star f^o \to \gamma \star fg^o \text{ where } g \in \mathcal{C}(c, c')$$

then $H(\gamma; -)$ is a representable functor with c_γ as the representing object.

Note that $H(\gamma; -)$ and $\mathcal{C}(c_\gamma; -)$ are naturally isomorphic and the natural isomorphism η_γ is given by $\eta_\gamma(c_\gamma)\colon H(\gamma; c_\gamma) \to \mathcal{C}(c_\gamma, c_\gamma)$ sending $\gamma \to I_{c_\gamma}$.

Proposition 3.4. *For $\gamma, \gamma' \in \mathcal{PC}$, we have the following.*

1. *$H(\gamma; -) \subseteq H(\gamma'; -)$ if and only if there exists a unique epimorphism h from c_γ to c_γ such that $\gamma = \gamma' \star h$.*
2. *If $R_\gamma \leq R'_\gamma$ then $H(\gamma; -) \subseteq H(\gamma'; -)$ and the converse holds only when γ, γ' are normal cones.*

Proof. 1. Let $\gamma = \gamma' \star h$ where $h\colon c'_\gamma \to c_\gamma$ is an epimorphism and $c \in v\mathcal{C}$. If $\gamma \star f^o \in H(\gamma; c)$ we have $\gamma \star f^o = \gamma' \star hf^o$. Since h is an epimorphism by the uniqueness of

canonical factorization $(hf)^o = hf^o$ and so $\gamma \star f^o \in H(\gamma'; c)$. Thus $H(\gamma; c) \subseteq H(\gamma'; c)$ for all $c \in v\mathcal{C}$. Let $g\colon c \to c'$ be a morphism. Then $H(\gamma; g)(\gamma \star f^o) = \gamma \star fg^o$ and

$$H(\gamma'; g)(\gamma \star f^o) = H(\gamma'; g)(\gamma' \star (hf)^o) = \gamma \star (fg)^o$$

and the following diagram commutes:

$$\begin{tikzcd} H(\gamma'; c) \arrow[r, "H(\gamma';g)"] & H(\gamma'; c') \\ H(\gamma; c) \arrow[u, "j_{H(\gamma;c)}^{H(\gamma';c)}"] \arrow[r, "H(\gamma;g)"'] & H(\gamma; c') \arrow[u, "j_{H(\gamma;c')}^{H(\gamma';c')}"'] \end{tikzcd}$$

hence $H(\gamma; -) \subseteq H(\gamma'; -)$

Conversely, suppose that $H(\gamma; -) \subseteq H(\gamma'; -)$. Then $\gamma \in H(\gamma; c_\gamma) \subseteq H(\gamma'; c_\gamma)$ and so $\gamma = \gamma' \star f^o$ for some $f \in \mathcal{C}(c_{\gamma'}, c_\gamma)$. $c_{\gamma' \star f^o} = imf$ so $imf = c_\gamma = codf$. Hence it follows that f is an epimorphism and $\gamma = \gamma' \star f$. If $\gamma' \star h = \gamma' \star k$, then for any $c \in N'_\gamma$ we have $\gamma'(c)h = \gamma'(c)k$ and since $\gamma'(c)$ is an epimorphism $h = k$. 2. If $R_\gamma \leq R_{\gamma'}$, then $\gamma = \gamma' \cdot \tau$ for some $\tau \in \mathcal{PC}$ and so $\gamma = \gamma' \star h$ for some epimorphism. Hence by (1),$H(\gamma; -) \subseteq H(\gamma'; -)$ and $H(\gamma; -) = H(\gamma'; -)$ when γ, γ' are normal cones □

Corollary 3.1. *Let $\gamma, \gamma' \in \mathcal{PC}$. If $H(\gamma; -) \subseteq H(\gamma'; -)$ then $N_\gamma \subseteq N_{\gamma'}$.*

Proof. Let $c \in N_{\gamma'}$ then $\gamma'(c)$ is an epimorphism. $H(\gamma; -) \subseteq H(\gamma'; -)$ implies $\gamma = \gamma' \star h$ where $h\colon c_{\gamma'} \to c_\gamma$ is an epimorphism and $\gamma(c) = \gamma'(c)h$ is epi. Hence $c \in N_\gamma$ and $N_{\gamma'} \subseteq N_\gamma$. □

Remark 3.1. *Let $\mathcal{C}$ be a proper category and $\gamma, \gamma' \in \mathcal{PC}$. Then*
1. $L_\gamma \leq L'_\gamma \Rightarrow c_\gamma \subseteq c'_\gamma$
2. $R_\gamma \leq R'_\gamma \Rightarrow H(\gamma; -) \subseteq H(\gamma'; -)$.

4 Ideals Categories Of Semigroups

In the following we proceed to describe the categories of left [right] ideals of a semigroup. Let S be a semigroup and S^1 the semigroup obtained by adjoining 1 to S (if S has no 1). $\mathbb{L}(S)$ is the category with

$$v\mathbb{L}(S) = \{S^1 a\colon a \in S^1\} \qquad \text{and for a,b} \in S^1$$

$$\mathbb{L}(S)(S^1 a, S^1 b) = \{\rho(a, s, b) = \rho_s | S^1 a\}$$

where $\rho(a, s, b)\colon x \to xs$ with $as \in S^1 b; x \in S^1 a$

$$(st)\rho = s(t\rho) \text{for all s,t} \in S^1 a$$

It is easy to observe that $\mathbb{L}(S)$ is a category with composition of morphisms given by

$$\rho(a, s, b) \cdot \rho(c, t, d) = \begin{cases} \rho(a, st, d) \text{ if } S^1 b = S^1 c \\ undefined\, otherwise. \end{cases}$$

Proposition 4.1. *Let $\rho(a, s, b)\colon S^1 a \to S^1 b$ be a morphism in $\mathbb{L}(S)$. Then*

1. *$\rho(a, s, b)$ is epimorphism if and only if $as\mathcal{L}b$*
2. *$\rho(a, s, b)$ is a split monomorphism if and only if $a\mathcal{R}as$*
3. *$\rho(a, s, b)$ is an isomorphism if and only if $a\mathcal{R}as\mathcal{L}b$*

Proof. It is easy to observe that $\rho(a, s, b)$ is epi if and only if $S^1 as = S^1 b$ and so $as\mathcal{L}b$. Suppose $a\mathcal{R}as$ then there exists $u, v \in S^1$ such that $au = as$ and $asv = a$. Hence there exists $\sigma(b, v, a)$ such that $\rho(a, s, b)\sigma(b, v, a) = I_{S^1 a}$ and $\rho(a, s, b)$ is split inclusion. Conversely suppose $\rho(a, s, b)$ is a split monic then there is a $\sigma(b, s', a)\colon S^1 b \to S^1 a$ such that $\rho\sigma = I_{S^1 a}$ which implies $asv = a$ and $a\mathcal{R}as$. □

Proposition 4.2. *$\mathbb{L}(S)$ is a category with sub objects, every morphism has unique canonical factorization.*

Proof. If $\rho(a, s, b) = \rho(a, 1, b)$ where $a \cdot 1 \in S^1 b$, then $\rho(a, s, b) = j_{S^1 a}^{S^1 b}$. Moreover $\{\rho(a, 1, b)\colon a \in S^1 b\}$ is a choice of sub objects in the category $\mathbb{L}(S)$. If $\rho(a, s, b)$ is a morphism in $\mathbb{L}(S)$ then $\operatorname{Im} \rho(a, s, b) = S^1 as$ and $\rho(a, s, b) = \rho(a, s, as) \cdot \rho(as, 1, b)$ gives the image factorization of $\rho(a, s, b)$ in $\mathbb{L}(S)$. □

Proposition 4.3. *$\mathbb{L}(S)$ is a category with subobjects $\rho^d\colon v\mathbb{L}(S) \to \mathbb{L}(S)$ is a cone with vertex $S^1 d$ defined by $\rho^d(S^1 a) = \rho(a, s, d)$ where $as \in S^1 d$ is a proper cone.*

Proof. $\rho^d(S^1 a) \in \mathbb{L}(S)(S^1 a, S^1 d)$ is well defined. If $S^1 a \subseteq S^1 b$ then

$$\begin{aligned} j_{S^1 a}^{S^1 b}\rho^d(S^1 b) &= \rho(a, 1, b) \cdot \rho(b, v, d) \text{ where bv = qd} \in S^1 d\,. \\ &= \rho(a, v, d)\,. \end{aligned}$$

Since $S^1 a \subseteq S^1 b$, $a = rb$ for some $r \in S^1$ and $av = rbv = rqd \in S^1 d$. Any element of $\rho^d(S^1 a)$ is of the form $\rho(a, v, d)$ where $av = rbv = rqd \in S^1 d$, thus $j_{S^1 a}^{S^1 b} \cdot \rho^d(S^1 b) = \rho^d(S^1 a)$. Since $\rho(a, d, d)$ is an epimorphism, it follows that ρ^d admits an epimorphic component and hence $\rho^d\colon v\mathbb{L}(S) \to \mathbb{L}(S)$ is a proper cone. □

For any semigroup S, $\mathbb{L}(S)$ is a proper category, the map $\rho^d(S^1 a) = \rho(a, s, d)$ is proper, hence the cone ρ^d in $\mathbb{L}(S)$ is with vertex $S^1 as = S^1 d$ and

$$N_{\rho^d} = \{S^1 a | as\mathcal{L}d\}\,.$$

For every $S^1 a \in v\mathbb{L}(S)$ there is a cone ρ^a with $\rho^a(S^1 a) = I_{S^1 a}$. In the following we denote by $\mathcal{P}\mathbb{L}(S)$ the semigroup of all proper cones in $\mathbb{L}(S)$.

Remark 4.1. *If S^{op} denote the opposite semigroup of S with multiplication given by $a \circ b = b \cdot a$ where the right side is the product in S then $\mathcal{L}(S^{op}) = \mathcal{R}(S)$ and $\mathcal{R}(S^{op}) = \mathcal{L}(S)$.*

Thus for any statement which holds for $\mathcal{L}(S)$ (or $\mathcal{R}(S)$) the corresponding dual statement holds for $\mathcal{R}(S)$ (respectively $\mathcal{L}(S)$). Hence $\mathbb{R}(S)$ is also a proper category with

$$v\mathbb{R}(S) = \{aS^1 : a \in S^1\} \text{ and for } a, b \in S^1 , \tag{1}$$

$$\mathbb{R}(S)(aS^1, bS^1) = \{\lambda(a, s, b) = \lambda_s | aS^1 : s \in S^1 \text{ with } sa \in bS^1\} \tag{2}$$

4.1 Representation of *S* by proper cones

Right regular representation of the semigroup S is the homomorphism $\rho : a \to \rho_a$ of S into the full transformation semigroup $\mathcal{T}_S$. Let S_ρ be the image of ρ so that $\rho : S \to S_\rho$ is a surjective homomorphism. S is said to be right reductive if ρ is injective.

Theorem 4.1. *Let S be a semigroup and $\mathbb{L}(S)$ is the proper category. Then there exists a homomorphism $\bar{\rho} : S \to \mathcal{P}\mathbb{L}(S)$ and an injective homomorphism $\phi : S_\rho \to \mathcal{P}\mathbb{L}(S)$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\rho"] \arrow[d] & S_\rho \\ S \arrow[r, "\bar{\rho}"'] & \mathcal{P}\mathbb{L}(S) \arrow[u, "\phi"'] \end{tikzcd}$$

In particular S is isomorphic to a subsemigroup of $\mathcal{P}\mathbb{L}(S)$ if and only if S is right reductive.

Proof. $\mathbb{L}(S)$ is a proper category and for each $Sa \in v\mathbb{L}(S)$, ρ^a is a proper cone with vertex Sa. For $a \in S$ define $\phi(\rho_a) = \rho^a$, if $\rho_a = \rho_b$ then $Sa = Sb$ and for $x \in Sd$ we have

$$\begin{aligned} x\rho^a(Sd) &= x\rho(d, u, a) = xu \text{ ; where } du \in Sa = Sb \\ &= x\rho(d, u, b) = x\rho^b(Sd) \end{aligned}$$

ie., $\rho^a = \rho^b$. Similarly if $\rho^a = \rho^b$ then for all $x \in S$, $xa = xb$ and so $\rho_a = \rho_b$, hence $\phi : S_\rho \to \mathcal{P}\mathbb{L}(S)$ is injective. Now for $a, b \in S$ and $Sd \in v\mathbb{L}(S)$ we have

$$\begin{aligned} \rho^a \cdot \rho^b(Sd) &= \rho^a(Sd) \cdot \rho^b(Sa)^o \\ &= \rho(d, a, a) \cdot \rho(a, b, b)^o \\ &= \rho(d, a, a) \cdot \rho(a, b, ab) \\ &= \rho(d, ab, ab) \\ &= \rho^{ab}(Sd) \end{aligned}$$

Hence $\phi(\rho_a)\phi(\rho_b) = \phi(\rho_{ab})$ and ϕ is an injective homomorphism. If we set $\bar{\rho} = \rho\phi$ then $\bar{\rho} : S \to \mathcal{P}\mathbb{L}(S)$ is a homomorphism making the above diagram commute. The last statement follows from the fact that $\bar{\rho}$ is injective if and only if ρ is injective. □

Two proper categories are isomorphic if there is an isomorphism that preserves inclusions. Note that if S is a semigroup then $\mathbb{L}(S)$ is a proper category. Now we proceed to show that every proper category arises in this way.

Theorem 4.2. *Let $\mathcal{C}$ be a proper category. Define $F\colon \mathcal{C} \to \mathbb{L}(P\mathcal{C})$ by*

$$vF(c) = \mathcal{P}\mathcal{C}\gamma$$
$$F(f) = \rho(\gamma, \gamma \star f^o, \gamma')$$

where $\gamma, \gamma' \in \mathcal{P}\mathcal{C}$ with $c_\gamma = c,\ c_{\gamma'} = d$ and $f \in \mathcal{C}(c, d)$. Then F is a faithfull functor.

Proof. The map vF defined above is an order isomorphism of $v\mathcal{C}$ with the partially ordered set $v\mathbb{L}(\mathcal{P}\mathcal{C})$ of all principal left ideals of $\mathcal{P}\mathcal{C}$. Since $\mathcal{C}$ is proper given any $f \in \mathcal{C}(c, d)$, there exists $\gamma, \gamma' \in \mathcal{P}\mathcal{C}$ with $c_\gamma = c$ and $c_{\gamma'} = d$. Since the epimorphic component f^0 is unique, $F(f) = \rho(\gamma, \gamma \star f^o, \gamma')$ ie well defined. To show that F is a functor, consider $f, g \in \mathcal{C}$ with $F(f) = \rho(\gamma, \gamma \star f^o, \gamma')$ and $F(g) = \rho(\gamma_1, \gamma_1 \star g^o, \gamma'')$ then fg exists in $\mathcal{C}$ if and only if $\gamma' \mathcal{L} \gamma_1$ and hence

$$F(f)F(g) = \rho(\gamma, (\gamma \star f^o) \cdot (\gamma' \star g^o), \gamma'') \,.$$

If $c_1 = imf = imf^o$ then

$$\begin{aligned}(\gamma \star f^o) \cdot (\gamma' \star g^o) &= (\gamma \star f^o) \star ((\gamma' \star g^o)(c_1))^o \\ &= (\gamma \star f^o) \star (j^d_{c_1} g^o)^o since c_1 \subseteq c = c_{\gamma'} \\ &= (\gamma \star f^o) \star (j^d_{c_1} g)^o \\ &= (\gamma \star (fg)^o)\end{aligned}$$

hence $F(f)F(g) = F(fg)$. From the definition of F it is clear that if $c' \subseteq c,\ c_\gamma = c$ and $c_{\gamma'} = c'$ then

$$F(j^c_{c'}) = \rho(\gamma', \gamma', \gamma) = j^{(\mathcal{P}\mathcal{C})\gamma}_{(\mathcal{P}\mathcal{C})\gamma'}$$

thus F is an inclusion preserving functor.

Since every morphism in $\mathcal{C}$ has a unique canonical factorization the map $f \to f^o$ is a bijection of $\mathcal{C}(c, d)$ onto the set of epimorphisms h with $dom\, h = c$ and $cod\, h \subseteq d$. Since every cone in $\gamma(\mathcal{P}\mathcal{C})\gamma'$ is uniquely representable in the form $\gamma \star f^o$ where $f \in \mathcal{C}(c_\gamma, c_{\gamma'})$ there is an injection between $\mathcal{C}(c_\gamma, c_{\gamma'})$ and $\gamma(\mathcal{P}\mathcal{C})\gamma'$. Hence the map $f \to \rho(\gamma, \gamma \star f^o, \gamma')$ is a injection of $\mathcal{C}(c_\gamma, c_{\gamma'})$ onto $\mathcal{L}(\mathcal{P}\mathcal{C})(\mathcal{P}\mathcal{C}\gamma, \mathcal{P}\mathcal{C}\gamma')$ and hence F is fully faithful. ☐

5 Proper Dual

If $\mathcal{C}$ is a proper category, then the proper dual of $\mathcal{C}$ denoted by $\mathbb{P}^\star\mathcal{C}$ is the full subcategory of $\mathcal{C}^\star$ with

$$v\mathcal{P}^\star\mathcal{C} = \{H(\gamma; -)\colon \gamma \in \mathcal{P}\mathcal{C}\} \,.$$

Lemma 5.1 (cf.[2]). *To every morphism $\sigma\colon H(y;-) \to H(y';-)$ in $\mathbb{P}^\star\mathcal{C}$ there is a unique $\hat{\sigma}\colon c_{y'} \to c_y$ in $\mathcal{C}$ such that the following diagram commutes.*

$$\begin{tikzcd} H(y';-) \arrow[r, "\eta_{y'}"] & \mathcal{C}(c_{y'};-) \\ H(y;-) \arrow[u, "\sigma"] \arrow[r, "\eta_y"'] & \mathcal{C}(c_y;-) \arrow[u, "\mathcal{C}(\hat{\sigma};-)"'] \end{tikzcd}$$

In this case, the component of the natural transformation σ at $c \in v\mathcal{C}$ is given by

$$\sigma(c)\colon y \star f^o \to y' \star (\hat{\sigma}f)^o$$

In particular σ is the inclusion $H(y;-) \subseteq H(y';-)$ if and only if

$$y = y' \star \hat{\sigma} \,.$$

Moreover the map $\sigma \to \hat{\sigma}$ is a bijection of $\mathbb{P}^\star\mathcal{C}(H(y;-), H(y';-))$ onto $\mathcal{C}(c_y, c_{y'})$.

Lemma 5.2. *If $f\colon c \to d$ is an epimorphism in $\mathcal{C}$, then $H(y;f)\colon H(y;c) \to H(y;d)$ in $\mathcal{P}^\star\mathcal{C}$ is an epimorphism.*

Proof. Let $f\colon c \to d$ is an epimorphism. Consider $H(y;c) = \{y \star h^o \mid h\colon c_y \to c\}$, $H(y;d) = \{y \star (hf)^o\}$ and

$$H(y;f)(y \star h^o) = y \star (hf)^o \,.$$

For $k_1, k_2\colon c \to a$, $H(y;k_1), H(y;k_2)\colon H(y;d) \to H(y;a)$ and for every $y \star h^o \in H(y;c)$, $H(y;f)H(y;k_1)(y \star h^o) = H(y;f)H(y;k_2)(y \star h^o)$

$$\begin{aligned} H(y;k_1)(H(y;f)(y \star h^o)) &= H(y;k_2)(H(y;f)(y \star h^o)) \\ H(y;k_1)(y \star (hf)^o) &= H(y;k_2)(y \star (hf)^o) \\ H(y;k_1) &= H(y;k_2) \end{aligned}$$

□

Remark 5.1. *Dually if $f\colon c \to d$ is a monomorphism in $\mathcal{C}$ then $H(y;f)\colon H(y;c) \to H(y;d)$ in $\mathcal{P}^\star\mathcal{C}$ is also a monomorphism.*

Theorem 5.1. *For any proper category $\mathcal{C}$ the dual category $P^\star\mathcal{C}$ is also a proper category.*

Proof. Since inclusion $j\colon c \subseteq d$ splits in $\mathcal{C}$ there exists retraction $e\colon d \to c$, such that $je = I_c$. The inclusion $H(y;j)(y \star f^o) = y \star (fj)^o$ where $f\colon c_y \to c$ splits since $H(y;e)(y \star g^o) = y \star (ge)^o$ where $g = fj\colon c_y \to d$ and $H(y;j)H(y;e) = H(y;I_c) = I_{H(y;c)}$. That is $H(y;j)\colon H(y;c) \subseteq H(y;d)$ splits in $\mathcal{P}^\star C$. $\sigma\colon H(y;-) \subseteq H(y';-)$ is an inclusion if and only if $y = y' \star \hat{\sigma}$ where $\hat{\sigma}\colon c_{y'} \to c_y$ in $\mathcal{C}$. Then $\sigma(c)\colon H(y;c) \subseteq H(y';c')$ has a right inverse $\sigma(c)^{-1}$ such that

$$\{(y' \star hf^\circ)\}\sigma(c)^{-1} = \{(y \star f^\circ)\sigma(c)\} \,.$$

Let $f\colon c \to d$ in $\mathcal{C}$ has a unique canonical factorization $f = qj$ where $q\colon c \to a$ and $j\colon a \subseteq d$ then $H(\gamma; f)\colon H(\gamma; c) \to H(\gamma; d)$ in $\mathcal{P}^{\star}\mathcal{C}$ maps

$$H(\gamma; f)\colon \gamma \star h^{o} \to \gamma \star (hf)^{o} \text{ where } h\colon c_{\gamma} \to c$$

$$\begin{aligned} H(\gamma; q)H(\gamma; j)(\gamma \star h^{o}) &= H(\gamma; j)(H(\gamma; q)(\gamma \star h^{o})) \\ &= H(\gamma; j)(\gamma \star (hq)^{o}) \\ &= (\gamma \star (hqj)^{o}) \\ &= (\gamma \star (hf)^{o}) \\ &= H(\gamma; f)(\gamma \star h^{o}) \end{aligned}$$

Thus $H(\gamma; f) = H(\gamma; q)H(\gamma; j)$, where $H(\gamma; q)$ is an epimorphism and $H(\gamma; j)$ is an inclusion. □

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On Nakayama Conjecture and related conjectures-Review

Abstract: In this paper, we review the "Nakayama Conjecture" and relating problems so called "Generalized Nakayama Conjecture", "Strong Nakayama Conjecture", "Tachikawa Conjecture", "Finitistic dimension Conjecture"and we give other conjectures proposed recently. We discuss relations between their conjectures.

Keywords: Nakayama Conjecture; dominant dimension; QF-3 ring; QF-ring.

1 Introduction

The following Nakayama Conjecture is the old and world wide famous conjecture proposed in [11] by Japanese mathematician "Tadashi Nakayama" in 1958 who was the professor in Nagoya University.

Nakayama Conjecture: *Let A be a finite dimensional algebra over a field K and* $0 \to A \to E_1 \to E_2 \to \ldots$ *a mininal injective resolution of A. If all E_i's are projective, then A is self-injective.*

This conjecture was his challenge to homological algebra, in fact, he suspected the power of homological algebra which had been created in this era.
In fact, Professor Hiroyuki Tachikawa taught us in his last lecture in 1994 that Professor Nakayama had proposed;

"If homological algebra is enough powerful to mathematics, then solve this problem."

Since then, many many ring theorists attempt to solve Nakayama Conjecture. It is no doubt that the greatest contributor of studying of Nakayama Conjecture is Tachikawa. Let's consider one aspect of Nakayama Conjecture, like the injective hull $E(R)$ of a ring R is projective. A ring with this property is called QF-3 ring. Every one knows that theory of QF-3 rings is very important in Ring Theory. The theory of QF-3 rings is developed and summarized in Tachikawa's lecture note [13] published in 1973

There are two purposes in this paper. One is to shed light on Nakayama Conjecture again by summarizing various aspect s of Nakayama Conjecture. The other is that ring theorists in the world are interested in Nakayama Conjecture and we anticip-

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ate that they solve Nakayama Conjecture and related conjectures to progress ring theory more actively. The author believes that attempts to challenging these conjectures create important notions and theories of rings with homological algebras as done by Tachikawa.

In this paper, we give some relating conjectures and discuss their relations. The following is the list of contents in this paper.
Section 2. Nakayama Conjecture
Section 3. Tachikawa Conjecture +
Section 4. Generalized Nakayama Conjecture
Section 5. Strong Nakayama Conjecture
Section 6. Finitistic Dimension Conjecture
Section 7. Tilting version of Generalized Nakayama Conjecture
Section 8. Related Results

2 Nakayama Conjecture

Let A be a finite dimensional algebra over a field K and $D(M) = \mathrm{Hom}_k(M, K)$ a dual space of a vector space M. Nakayama gave the following conjecture in 1958 [11].

Conjecture 1 (NC: Nakayama Conjecture). *Assume ${}_AA$ has a minimal injective resolution*

$$0 \to A \to E_1 \to E_2 \to \cdots \to E_n \to \cdots$$

such that all E_i's are projective, then A is self-injective.

Tachikawa gave the following conjecture which is equivalent to [NC]. (See Theorem 2.7.)

Conjecture 2 (TC: Tachikawa Conjecture).
[T1] $\mathrm{Ext}^i_A({}_AD(A), {}_AA) = 0$ *for all $i > 0$, then A is self-injective.*
[T2] *Assume A is a self-injective algebra and M is a finitely generated left A module. If* $\mathrm{Ext}^i_A(M, M) = 0$ *for all $i > 0$, then M is projective.*

Remark 2.1. [T2] *and hence* [NC] *are not true for an artinian ring in general. We see this in section* 8(7). *This means that* [NC] *is a typical conjecture for algebras.*

We consider what is the difference between algebras and artinian rings. One aspect is that it has duality or not. In general, an artinian ring has not self-duality, so we give the following new conjecture.

Conjecture 3 (NNC: New Nakayama Conjecture). *Assume an artinian ring A has a self-duality and ${}_AA$ has a minimal injective resolution*

$$0 \to A \to E_1 \to E_2 \to \cdots \to E_n \to \cdots$$

such that all E_i's are projective, then A is self-injective.

Typical example of an artinian ring with self-duality is an *artin algebra*, which is an artinian ring finitely generated over its center. An artin algebra was orginally defined by Emil Artin. For details, refer to Maurice Auslander, Idun Reiten, Sverre O. Smalo's book [3].

The following rings are typical rings with self-duality (Yoshitomo Baba [4]).

(1) Commuative ring

(2) Serial ring (Amdal, Ringdal,1968 [1])

(3) Harada(H) ring with homogenious socle (i.e. soc$R$ is a finite direct sum of a simple module.)

A ring R is called left H-*ring* if for any indecomposable projective right module P_R, there is some indecomposable projective injective right module I such that $P = I rad^n R$ for some $n > 0$.

(4) Homogenius type Harada ring (Jiro Kado and Kiyoichi Oshiro[8]).

(5) Some Quasi-Harada(QH) rings

Definition 2.2. 1. *A ring R is called* QH-*ring if any projective left (right) module is quasi-injective.*

2. *A ring A is called a left* QF-3 *ring if it satisfies the one of the following equivalent conditions;*
   (a) $E(A) \subset \prod A$, *here $E(A)$ is an injective envelop of ${}_AA$.*
   (b) *A has a mininaml faithful module ${}_AM$.*
   (i.e.) *${}_AM$ is faithful and for any faithful module ${}_AN$, it holds $N\oplus > M$.*
   (c) *There is an idempotent $f = f^2 \in A$ such that Af is faithful injective.*
3. *A ring A is called* QF-3 *ring if A is a left and right QF-3 ring.*

QH rings are considered the general notion of QF-3 rings by the following theorem.

Theorem 2.3. QH *ring is* QF-3 *ring.*
If eRe is local serial and (center of eRe)$\cap$(eradRe – (eradRe)^2) is not empty, then R has self-duality.

The following is an example of a ring of the above theorem.

Example. *Let D be a division ring and set $R = D \times D \times D$ with the multiplication*

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1y_1, x_1y_2 + x_2y_1, x_1y_3 + x_2y_2 + x_3y_1) .$$

Then R is non-commutative local serial ring with loewy length 3 and $(0, 1, 0)$ is in (center of eRe) $\cap$ (e(rad$R$)e – (e(rad$R$)e)^2).

To show the equivalence of [NC] and [TC], it requires the following facts and notations.

Lemma 2.4. *It holds for finite dimensional algebras over a field K*

$$\mathrm{Ext}^i_A({}_AD(A), {}_AA) \cong \mathrm{Ext}^i_{A^e}(A, A^e) .$$

Here, $A^e = A \otimes_K A^{op}$ is an enveloping algebra of A.

Proof.

$$\begin{aligned}\operatorname{Ext}^i_A(D(A_A), {}_AA) &= \operatorname{Ext}^i_{A\otimes_K K}({}_AA \otimes_A D(A_A)_K, {}_AA_K)\\ &\cong \operatorname{Ext}^i_{A^e}({}_AA_A, \operatorname{Hom}_K(D(A_A)_K, {}_AA_K))\ .\end{aligned}$$

Also,

$$\begin{aligned}{}_A\operatorname{Hom}_K(D(A_A)_K, {}_AA_K)_A &= {}_A\operatorname{Hom}_K(D(A_A)_K, D({}_KD({}_AA_K)))_A\\ &\cong {}_A\operatorname{Hom}_K(D(A_A) \otimes_K D({}_AA_K), {}_KK)_A\\ &\cong D(D({}_AA \otimes_K A_A))\\ &\cong {}_AA \otimes_K A_A\ .\end{aligned}$$

□

Definition 2.5. (*left dominant dimension*)
We denote ℓ.dom.dim$A \geq n$ *when a ring* A *has a minimal injective resolution*

$$0 \to A \to E_1 \to E_2 \to \cdots \to E_n \to \cdots$$

with projective modules $E_1, \ldots, E_n$.

Lemma 2.6 ([13] p.p.97). *Let* A *be a* QF-3 *ring with minimal faithful modules* Ae *and* fA. *Assume* ℓ.dom.dim $A \geq 2$ *and the first* n *images of the minimal injective resolution of* ${}_{fAf}fA$ *are finitely cogenerated by* ${}_{fAf}fAe$, *then the the following conditions are equivalent.*

1. ℓ.dom.dim $A \geq n+2$.
2. $\operatorname{Ext}^i_{fAf}(fA, fA) = 0$ *for* $i = 1, 2, \ldots, n$.
3. $\operatorname{Ext}^i_{eAe}(Ae, Ae) = 0$ *for* $i = 1, 2, \ldots, n$.

Now we give the proof that [NC] and [TC] are equivalent.

Theorem 2.7. [NC] $\iff$ [TC]

Proof. Assume [NC]. We first prove [T1]. We set $R = \operatorname{End}_A(A \oplus D(A))$. Let f and e in R be projections to A and $D(A)$, respectively. Then it holds $fRf = A$, $fR = fRf \oplus fRe = A \oplus D(A)$ as left A-module. Since

$$\begin{aligned}\operatorname{Ext}^i_{fRf}(fR, fR) &= \operatorname{Ext}^i_A(A \oplus D(A), A \oplus D(A))\\ &= \operatorname{Ext}^i_A(D(A), A)\ ,\end{aligned}$$

we have $\operatorname{Ext}^i_{fRf}(fR, fR) = 0$ from [T1]. By Lemma 2.6, we know ℓ.dom.dim.$R = \infty$. So R is self-injective by [NC]. Thus A is also self-injective. (See Lemma 2.8 below.)

Next we prove [T2]. Assume A is self-injective and M is finitely generated. We set $R = \operatorname{End}_A(A \oplus M)$. Let f and e in R be projections to A and M, respectively. By the same argument in the proof of [T1], it holds $\operatorname{Ext}^i_{fRf}(fR, fR) = \operatorname{Ext}^i_A(M, M) = 0$ and R is self-injective.

On the other hand, since $A \oplus M$ is finitely generated generator (cogenerator), it is well known that this satisfies double centralizer property. i.e. $\operatorname{End}_R(A \oplus M) = A$. Hence $A \oplus M$ is a projectice A-module. (See Lemma 2.8.) Thus M is projective. □

Lemma 2.8. 1. *Assume $_AM$ is finitely generated and* $\mathrm{Ext}^1_A(M, M) = 0$. *If* $R = \mathrm{End}_A(M)$ *is right self-injective, then M is a projective* $\mathrm{End}_R(M_R)$*-module.*

2. *Assume* $R = \mathrm{End}_A(A \oplus D(A))$ *is self-injective, then* $\mathrm{Ext}^1_A(D(A), A) = 0$ *iff A is self-injective.*

Proof. (1) We take a short exact sequence of left A-modules;

$$0 \to N \to \oplus A \to M \to 0\,.$$

We apply $\mathrm{Hom}_A(-, M)$ to the above exact sequence, we have the split short exact sequence of right R-modules

$$0 \leftarrow \mathrm{Hom}_A(N, M) \leftarrow \mathrm{Hom}_A(\oplus A, M) \leftarrow \mathrm{Hom}_A(M, M) = R \leftarrow 0$$

from the assumptions $\mathrm{Ext}^1_A(M, M) = 0$ and R is right self-injective.
We apply $\mathrm{Hom}_R(-, M_R)$ to the above exact sequence, we have the split exact sequence;

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_A(M, N), M) \to \oplus\,\mathrm{End}_R(M) \to \mathrm{Hom}_R(R, M) = M \to 0\,.$$

Thus M is a projective $\mathrm{End}_R(M)$-module.
(2) If part is clear, so we prove only if part. We remark $A = \mathrm{End}_R(M)$ since $_AM$ is generator. We apply (1) to $M = A \oplus D(A)$, then $_AD(A)$ is projective, that is, A_A is injective. So A is self-injective. ☐

Lemma 2.9. *Let $_AM$ be an A-module, $B = \mathrm{End}_A M$ and*

$$d\colon A \to \mathrm{End}_B M_B$$

a canonical map defined by $d(a)(m) = am$ for $a \in A$, and $m \in M$.

1. *d is a monomorphism iff $_AM$ is faithful.*
2. *If $_AM$ is generator, then d is an isomorphism and M_B is finitely generated projective.*
3. *If $_AM$ is finitely generated projective, then M_B is finitely generated generator.*

Proof. (1) is clear.
(2) Since generator is faithful, d is monomorphism. So we show d is an epimorphism. Take an epimorphism $\sum^n \oplus M \xrightarrow{(f_1, f_2, \dots, f_n)} A$, then there are some $m_j \in M$ $(j = 1, \dots, n)$ such that

$$1_A = f_1(m_1) + f_2(m_2) + \cdots + f_n(m_n)\,.$$

Also for $m \in M$, we define $\phi_m\colon {_AA} \to {_AM}$ by $\phi_m(a) = am$ for any $a \in A$. We remark $f_j\phi_m \in B$. For any $\varphi \in \mathrm{End}_B(M_B)$,

$$\varphi(m_j \cdot f_j\phi_{m_i}) = \varphi(m_j) \cdot f_j\phi_{m_i} = f_j(\varphi(m_j))m_i \in Am_i\,.$$

Since

$$\sum_{j=1}^n f_j(m_j)m_i = \Big(\sum_{j=1}^n f_j(m_j)\Big)m_i = m_i\,,$$

we have

$$\varphi(m_i) = \Big(\sum_{j=1}^{n} f_j(\varphi(m_j))\Big) m_i \in Am_i \,.$$

We set $\varphi(m_i) = a_i m_i$ and

$$a = a_1 f_1(m_1) + a_2 f_2(m_2) + \cdots + a_n f_n(m_n)\,,$$

then for any $m \in M$,

$$\begin{aligned} m = 1 \cdot m &= f_1(m_1)m + f_2(m_2)m + \cdots + f_n(m_n)m \\ &= m_1(f_1\varphi_m) + \cdots + m_n(f_n\varphi_m) \end{aligned}$$

So

$$\begin{aligned} \varphi(m) &= \varphi(m_1)f_1\varphi_m + \cdots + \varphi(m_n)f_n\varphi_m \\ &= (a_1 f(m_1) + \cdots + a_n f(m_n))m \\ &= am \end{aligned}$$

We apply $\mathrm{Hom}_A(-, {}_AM_B)$ to the above splittable epimorphism, then we have splittable epimorphism

$$\sum^{n} \oplus \mathrm{Hom}_A(M, {}_AM_B)_B = \sum^{n} \oplus B_B \to \mathrm{Hom}_A(A, {}_AM_B)_B = M_B \to 0\,.$$

Thus M_B is finitely generated projective.

(3) Assume ${}_AM$ is finitely generated projective, then we have a splittable epimorphim

$$\sum^{n} \oplus_A A \xrightarrow{(f_1, f_2, \ldots, f_n)} {}_AM \to 0\,.$$

That is, there are $f_i(1) = m_i \in M$ and $g_i \colon {}_AM \to {}_AA$ $(i = 1, \ldots, n)$ such that

$$m = m_1 g_1(m) + m_2 g_2(m) + \cdots + m_n g_n(m)$$

for any m. Hence $m = m_1(g_1\varphi_m) + \cdots + m_n(g_n\varphi_m)$. Remarking that $g_i\varphi_i \in B$, $m_1, \ldots, m_n$ are generators of M_B, that is, M_B is finitely generated B-module.

Apply $\mathrm{Hom}_A(-, {}_AM_B)$ to the above splittable exact sequence, we have a splittable epimorphism

$$\sum^{n} \oplus \mathrm{Hom}_A(A, {}_AM_B) = \sum^{n} \oplus M_B \to \mathrm{End}_A(M) = B_B \to 0\,.$$

That is, M_B is generator. □

3 Tachikawa Conjecture +

In the proof of Theorem 2.7, the properties of generator and co-generator are essential. So Tachikawa gave the following conjecture equivalent to [TC] by using the notion of generator and cogenerator.

Conjecture 4 (TC+: Tachikawa Conjecture +). *Let $_AM$ be finitely generated generator co-generator. If* $\mathrm{Ext}^i_A(M, M) = 0$ *for any* $i > 0$*, then M is projective.*

Theorem 3.1. [TC] $\Longleftrightarrow$ [TC+]

Proof. Assume [TC]. Since M is generator cogenerator, we have a splittable epimorphism $\sum \oplus M \to A \to 0$. That is, for some $m, n > 0$, it holds $_AA < \oplus M^{(n)}$ and $_AD(A) < \oplus M^{(m)}$. Thus $\mathrm{Ext}^i_A(M, M) = 0$ implies $\mathrm{Ext}^i_A(D(A), A) = 0$. A is self-injective by [T1] and M is projective by [T2].
Assume [TC+], then we have

$$0 = \mathrm{Ext}^i_A(D(A), A) = \mathrm{Ext}^i_A(D(A) \oplus A, D(A) \oplus A) .$$

We show [T1]. Since $D(A) \oplus A$ is projective,$A = D(D(A))$ is injective.
We show [T2]. $\mathrm{Ext}^i_A(M, M) = 0$ for $i > 0$ and A is self-injective implies $\mathrm{Ext}^i_A(M \oplus A, M \oplus A) = 0$. Also $D(A) \cong A$ implies $D(A)$ is generator cogenerator, thus $M \oplus D(A)$ is finitely generated generator cogenrator. By [TC+], $_AM$ is projective. □

4 Generalized Nakayama Conjecture

Mauris Auslnder and Idun Reiten gave the following conjecture in 1975 [2].

Conjecture 5 (GNC: Generalized Nakayama Conjecture). *Let* $0 \to A \to E_1 \to E_2 \to \cdots \to E_n \to \cdots$ *be a minimal injective resolution of* $_AA$ *and S any simple module. Then there is some i such that* $S < E_i$.

Remark 4.1. [GNC] $\Longleftrightarrow$ $\mathrm{Ext}^i_A(S, A) \neq 0$ *for some* $i > 0$

Conjecture 6 (GNC+: Generalized Nakayama Conjecture+). *A generator* $_AM$ *satisfying* $\mathrm{Ext}^i_A(M, M) = 0$ *for any* $i > 0$ *is finitely generated projective.*

Theorem 4.2. [GNC] $\Longleftrightarrow$ [GNC+]. *Particularly* [GNC] $\Longrightarrow$ [NC].

Proof. Assume [GNC]. We set $B = \mathrm{End}_A(M)$. Then M_B is finitely generated projective since $_AM$ is generator by Lemma 2.9(3). Let

$$0 \to {_AM} \to E_1 \to E_2 \to \cdots$$

be a minimal injective resolution of $_AM$.
We apply $\mathrm{Hom}_A(M, -)$, then the following sequence

$$0 \to B = {_B\mathrm{Hom}_A}({_AM_B}, {_AM}) \to {_B\mathrm{Hom}_A}({_AM_B}, E_1) \to {_B\mathrm{Hom}_A}({_AM_B}, E_2) \to \cdots$$

is exact since $\mathrm{Ext}^i_A(M, M) = 0$ for any $i > 0$. Also $_B\mathrm{Hom}_A({_AM_B}, {_AE_i})$ is injective since M_B is projective and $_AE_i$ is injective. Thus for some $m >> 0$, $\sum_{i=1}^{m} \oplus \mathrm{Hom}_A({_AM_B}, {_AE_i})$ is co-generator by [GNC].

On the other hand, ${}_AE_i < \oplus\sum^{t_i} \oplus D(A)$ since $D(A)$ is an injective cogenerator. So ${}_B\mathrm{Hom}_A({}_AM_B, E_i) < \oplus\sum^{t_i} \oplus {}_B\mathrm{Hom}_A({}_AM_B, D(A))$.

Since ${}_B\mathrm{Hom}_A({}_AM_B, D(A)) \cong {}_B\mathrm{Hom}_A(A \otimes_A M_B, A) = D(M_B)$ and $D(M_B)$ is co-generator, M_B is generator. Thus ${}_AM$ is finitely generated projective and [GNC+] holds.

We assume [GNC+]. Let

$$0 \to {}_AA \to E_1 \to E_2 \to \cdots$$

be a minimal injective resolution of ${}_AA$ and $\{S_1, S_2, \dots S_n\}$ the complete set of non-isomorphic simple modules included in some E_i.
We take $f \in A$ such that $f^2 = f$ and

$${}_AE(S_1) \oplus {}_AE(S_2) \oplus \cdots {}_AE(S_n) = {}_AD(fA)\ .$$

Then there is some m_i such that $E_i < \oplus_A D(fA)^{m_i}$. Remarking that $fA \otimes_A D(fA) = fD(fA) = D(fAf)$ as left fAf-module, we have natural isomorphisms

$$\begin{aligned} {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes_A D(fA)) &\cong {}_A\mathrm{Hom}_{fAf}(fA, D(fAf)) \\ &\cong {}_A\mathrm{Hom}_K(fAf \otimes_{fAf} fA_A, K) \\ &= {}_AD(fA_A)\ . \end{aligned}$$

Hence we have the natural isomorphism

$$\varphi_i\colon {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes E_i) \cong {}_AE_i\ .$$

From an exact sequence $0 \to {}_AA \to E_1 \to E_2$, we make an exact commutative diagram;

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_AA & \longrightarrow & E_1 & \longrightarrow & E_2 \\ & & & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} \\ 0 & \longrightarrow & {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes_A A) & \longrightarrow & {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes_A E1) & \longrightarrow & {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes_A E2). \end{array}$$

Since φ_1 and φ_2 are isomorphisms, we have an isomorphism

$${}_AA \cong \mathrm{End}_{fAf}(fA)\ .$$

On the other hand, $fA \otimes_A D(fA) = D(fAf)$ is an injective fAf-module, so is $fA \otimes_A E_i$. Hence we have an injective resolution of ${}_{fAf}fA = {}_{fAf}fA \otimes_A A$

$$0 \to fA \otimes_A A \to fA \otimes_A E_1 \to fA \otimes_A E_2 \to \cdots\ .$$

From the above two facts, we have $\mathrm{Ext}^i_{fAf}(fA, fA) = 0$. Hence ${}_{fAf}fA$ is finitely generated projective by [GNC+], so fA is a generator as left $\mathrm{End}_{fAf}(fA)(\cong A)$-module, that is, fA_A is a finitely generated projective generator. Thus ${}_AD(fA)$ is co-generator, which means $\{S_1, S_2, \dots S_n\}$ is the complete set of all non-isomorphic simple modules. hence [GNC] holds. □

5 Strong Nakayama Conjecture

Robert R. Colby and Kent R. Fuller gave the following conjecture in 1990 [5].

Conjecture 7 (SNC: Strong Nakayama Conjecture). *For any finitely generated module $_AM$, there is some $i \geq 0$ such that* $\mathrm{Ext}^i_A(M, A) \neq 0$.

Remark 5.1. *We can easily show that* [SNC] $\Longrightarrow$ [GNC].

6 Finitistic Dimension Conjecture

The *finitistic dimension* of an algebra A is defined by

$$f.\mathrm{gl.dim}A = \sup\{\mathrm{p.d}(M) < \infty\}$$

Here, $\mathrm{p.d}(M)$ is a projective dimension of $_AM$.

Conjecture 8 (FDC: Finitistic Dimension Conjecture).

$$f.\mathrm{gl.dim}A < \infty\,.$$

We show the following theorem.

Theorem 6.1.

$$[\mathrm{FDC}] \Longrightarrow [\mathrm{SNC}]$$

Proof. Assume $n = f.\mathrm{gl.dim}A < \infty$. Take $_AM$ such that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \geq 0$. Let

$$\cdots \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \xrightarrow{f_0} M \to 0$$

be a projective resolution of $_AM$. Then by assumption, we have an exact sequence

$$\cdots \leftarrow \mathrm{Hom}_A(P_1, A)_A \xleftarrow{\mathrm{Hom}_A(f_1,A)} \mathrm{Hom}_A(P_0, A)_A \xleftarrow{\mathrm{Hom}_A(f_0,A)} \mathrm{Hom}_A(M, A)_A = 0.$$

So we have the projective resolution of $\mathrm{ImHom}_A(f_{n+2}, A)$;

$$0 \leftarrow \mathrm{ImHom}_A(f_{n+2}, A) \xleftarrow{\mathrm{Hom}_A(f_{n+2},A)} \mathrm{Hom}_A(P_{n+1}, A)_A \leftarrow \cdots$$

$$\leftarrow \mathrm{Hom}_A(P_1, A)_A \xleftarrow{\mathrm{Hom}_A(f_1,A)} \mathrm{Hom}_A(P_0, A)_A \xleftarrow{\mathrm{Hom}_A(f_0,A)} \mathrm{Hom}_A(M, A)_A = 0.$$

Since $\mathrm{p.d\ ImHom}_A(f_{n+2}, A) \leq n$, we have a splittable epimorphism

$$0 \leftarrow \mathrm{Hom}_A(P_1, A)_A \xleftarrow{\mathrm{Hom}_A(f_1,A)} \mathrm{Hom}_A(P_0, A)_A\,.$$

Thus we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(\mathrm{Hom}_A(P_1, A)_A, A_A) \arrow[r, "g"] \arrow[d, "\cong"'] & \mathrm{Hom}_A(\mathrm{Hom}_A(P_0, A)_A, A_A) \arrow[r] \arrow[d, "\cong"'] & 0 \\
P_1 \arrow[r, "f_1"] & P_0 \arrow[r] & 0.
\end{tikzcd}$$

Here, $g = \mathrm{Hom}_A(\mathrm{Hom}_A(f_1 A_A)$. Thus f_1 is splittable epimorphism, which means $M = 0$. □

7 Tilting version of Generalized Nakayama Conjecture

Takayoshi Wakamatsu gave the following conjecture in his lecture which is equivalent to [GNC].

Conjecture 9 (TGNC: Tilting version of Generalied Nakayama Conjecture). *Assume T_A is a tilting module and let*

$$0 \to A \to T_1 \to T_2 \to \cdots \to T_n \to \cdots$$

be a minimal dominant resolution, then for any indecomposable direct summand $T' < \oplus T$, there is some i such that $T' < \oplus T_i$.

A module T_A is called a *tilting module* if the following two conditions are satisfied;
(1) $\mathrm{Ext}^i_A(T, T) = 0$ for any $i > 0$.
(2) There is some exact sequence

$$\cdots \to T_2 \to T_1 \to D(A)_A \to 0$$

such that $T_i < \oplus(\sum^{n_i} \oplus T)$ for every i and

$$\mathrm{Hom}_A(T, T_2) \to \mathrm{Hom}_A(T, T_1) \to \mathrm{Hom}_A(T, D(A)) \to 0\,.$$

is exact.

Let T_A be a tilting module. An exact sequence

$$0 \to A_A \to T_1 \to T_2 \to \cdots$$

is called a *dominant resolution* if the following two conditions are satisfied;
(1) $T_i < \oplus(\sum^{n_i} \oplus T_A)$ for every i.
(2) $0 \leftarrow \mathrm{Hom}_A(A, T) \leftarrow \mathrm{Hom}_A(T_1, T) \leftarrow \mathrm{Hom}_A(T_2, T) \leftarrow \cdots$ is exact.

Remark 7.1 (Wakamatsu). *There is a minimal dominant resolution.*

Wakamatsu proved the following theorem in his lecture.

Theorem 7.2. [GNC] $\iff$ [TGNC]

Proof. Assume [TGNC]. Let (*) $0 \to {}_AA \to {}_AI_1 \to {}_AI_2 \to \cdots$ be a minimal injective resolution of ${}_AA$. ${}_AD(A_A)$ is a tilting module with a minimal dominant resolution (*).

Indecomposable direct summands of ${}_A D(A_A)$ are injective envelops of all simple modules. That is, any simple module is a submodule of some I_i by [TGNC]. Hence [GNC] holds.

Next assume [GNC]. We set $B = \mathrm{End}_A(T_A)$. We know that a tilting module has the double centralizer property, we have $A = \mathrm{End}_B({}_B T)$. Let $0 \to A_A \to T_1 \to T_2 \to \cdots$ and $0 \to {}_B B \to T'_1 \to T'_2 \to \cdots$ be minimal dominant resolutions of A_A and ${}_B B$, respectively. We take a direct sum $\sum \oplus L$ of non-isomorphic indecomposable direct summands of some T_i. Since $\sum \oplus L < \oplus T$, there is $f \in B$ such that $f^2 = f$ and $\sum \oplus L = fT$. [TGNC] is equivalent to $f = 1_B$, so we show $f = 1_B$. We make a direct sum $\sum \oplus M$ of non-isomorphic indecomposable direct summands of some T'_i. By the same argument as above, there is $e \in A = \mathrm{End}(T_B)$ such that $e^2 = e$ and $\sum \oplus M = Te$. We know that

(1) ${}_{fAf} fTe_{eAe}$, ${}_B Bf_{fBf}$, ${}_{eAe} eA_A$ are tilting modules.

(2) ${}_B T_A \cong {}_B Bf \otimes_{fBf} fTe \otimes_{eAe} eA$.

(3) ${}_B T_A$ is a tilting module iff ${}_A \mathrm{Hom}_K(T, K)_B = D(T)$ is a cotilting module.

Since ${}_B Bf_{fBf}$ is a tilting module, there is an exact sequence

$$\cdots \to \sum \oplus Bf \to \sum \oplus Bf \to {}_B D(B) \to 0\ .$$

Hence we have an exact sequence

$$0 \to B \to \sum \oplus fD(B) \to \sum \oplus fD(B) \to \cdots\ ,$$

hence $f = 1_B$ by [GNC]. □

8 Related Results

We summarize the relation of each conjecture in the following implication for algebras.

[FDC] $\Longrightarrow$ [SNC] $\Longrightarrow$ [GNC] $\Longleftrightarrow$ [GNC+] $\Longleftrightarrow$ [TGNC] $\Longrightarrow$ [NC].

[NC] $\Longleftrightarrow$ [TC+] $\Longleftrightarrow$ [TC] $\Longleftrightarrow$ [TC1] and [TC2]

[NNC] for atinian rings $\Longrightarrow$ [NC] for algebras.

We give the cases that one of the conjectures hold.

1. **Theorem 8.1** (George V. Wilson [15]). [GNC] *is true for positive graded algebras.*

2. **Theorem 8.2** (Hiroyuki Tachikawa [13]). [T2] *is true for a group algebra $k[G]$ for a finite p-group G and a field k.*

3. **Theorem 8.3** (Rainer Schultz [12]). [T2] *is true for a group algebra $k[G]$ for a finite group G and a field k.*

4. **Theorem 8.4** (Edward L. Green, Birge Zimmermann-Huisgen [7]). [FDC] *is true for an algebra A with vanishing radical cube (i.e. $rad^3 A = 0$).*

5. **Theorem 8.5** (Peter Dräxler [6]). [GNC] *is true for algebras A with $rad^{2\ell+1}A = 0$ and $A/rad^{\ell}A$ representation finite.*

6. **Theorem 8.6** (Yong Wang [14]). [SNC] *is true for artinian rings R with $rad^{2\ell+1}R = 0$ and $A/rad^{\ell}R$ representation finite.*

   Proof. (Wang's proof). Assume there is finitely generated non-zero R-module ${}_RM$ such that $\mathrm{Ext}^i_R(M, R) = 0$ for all $i \geq 0$.
   For a projective resolution of M,

$$\cdots \to P_{n+1} \xrightarrow{f_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0\,,$$

   we set $\Omega_i = \mathrm{Im} f_i$ and denote $T^* = \mathrm{Hom}_R({}_RT, {}_RR_R)_R$.
   By assumption, we have an exact sequence

$$0 \leftarrow \Omega_i^* \leftarrow P_{i-1}^* \leftarrow \cdots \leftarrow P_1^* \xleftarrow{f_1^*} P_0^* \leftarrow 0.$$

   Thus p.d $\Omega_i^* \leq i-1$ for any $i \geq 1$. Since $\Omega_i^* \subset JP_i^*$, we have $J^{2\ell}\Omega_i^* = 0$ for any i. We prove $\mathrm{Ext}^1_R(\Omega_2^*, R) \neq 0$ and $\mathrm{Ext}^1_R(\Omega_i^*, R) = 0$ for any $i \geq 3$. Since $P \cong P^{**}$ for any projective module P, we have the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P_{n+1} & \xrightarrow{f_{n+1}} & P_n & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 \\
 & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\cdots \longrightarrow & P_{n+1}^{**} & \xrightarrow{f_{n+1}^{**}} & P_n^{**} & \longrightarrow & \cdots & \longrightarrow & P_1^{**} & \xrightarrow{f_1^{**}} & P_0^{**}.
\end{array}$$

   Thus

$$0 \longleftarrow (\Omega_{n+3})^* \longleftarrow (P_{n+2})^* \longleftarrow (P_{n+1})^* \longleftarrow \cdots$$

   is a projective resolution and

$$0 \longrightarrow (\Omega_{n+3})^{**} \longrightarrow (P_{n+2})^{**} \longrightarrow (P_{n+1})^{**}$$

   is exact, which means $\mathrm{Ext}^1_R((\Omega_{n+3})^*, R) = 0$ for any $n \geq 0$. Consider the following commutative diagram;

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Omega_2^{**} & \longrightarrow & P_1^{**} & \xrightarrow{f_1^{**}} & P_0^{**} & \longrightarrow & \mathrm{Ext}^1_R(\Omega_2^*, R) & \longrightarrow & 0 \\
 & & & & \cong\downarrow & & \cong\downarrow & & & & \\
 & & & & P_1 & \xrightarrow{f_1} & P_0 & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

   we know that f_1 is non-splittable. Thus $\mathrm{Ext}^1_R(\Omega_2^*, R) \neq 0$.

We fix $m \geq 1$. Take $0 \neq {}_RN$ such that p.d ${}_RN \leq m$, and $J^{2\ell}N = 0$. We set $N_1 = J^\ell N,\ N_2 = N/J^\ell N$.

Since R/J^ℓ is representation finite, let $\{C_1, \ldots, C_m\}$ be the complete set of non-isomorphic indecomposable modules and we have the decompositions

$$N_1 = \sum_{j=1}^{m} \oplus C_j^{a_j}\ N_2 = \sum_{j=1}^{m} \oplus C_j^{b_j}\ .$$

For $i > m$, $\mathrm{Ext}_R^{i+1}(N_1, R)_R \cong \mathrm{Ext}_R^i(N_2, R)$ is finitely generated.
We set $\ell(k, j) = \text{length}\, \mathrm{Ext}_R^k(C_j, R)_R$, then

$$\sum_{j=1}^{m} \ell(i+1, j) \cdot a_j = \sum_{j=1}^{m} \ell(i, j) \cdot b_j\ .$$

We denote $\mathbb{Z}$-module L_i $(i > m)$ by

$$\left\{(c_1, \ldots, c_m, d_1, \ldots, d_m) \in \mathbb{Z}^{2m} \;\middle|\; \sum_{j=1}^{m} \ell(i+1, j) \cdot c_j = \sum_{j=1}^{m} \ell(i, j) \cdot d_j \right\}.$$

They are $\mathbb{Z}$-submodules of the noetherian module $\mathbb{Z}^{2m}$. So an increasing sequence $L_0 \subset L_1 \subset \ldots$ terminates. That is, $L_{m_0} = L_{m_0+1} = \ldots.$ for some m_0. Take $N = (\Omega_{m_0+3})^*$. Remarking that p.d $(\Omega_{m_0+3})^* < m_0 + 2$, $(a_1, \ldots, a_m, b_1, \ldots, b_m) \in L_{m_0+2}$, thus $(\star)$ $(a_1, \ldots, a_m, b_1, \ldots, b_m) \in L_{m_0} = L_{m_0+1}$.
From the exact sequence

$$0 \to J^\ell N \to N \to N/J^\ell N \to 0$$

and the fact

$$\mathrm{Ext}_R^{m_0+1}((\Omega_{m_0+3})^*, R) \cong \mathrm{Ext}_R^1(\Omega_3^*, R) = 0\ ,$$

we have an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}_R^{m_0+1}(J^\ell N, R) \to \mathrm{Ext}_R^{m_0+2}(N/J^\ell N, R) \to \mathrm{Ext}_R^{m_0+2}(N, R) \to \\ &\mathrm{Ext}_R^{m_0+2}(J^\ell N, R) \to \mathrm{Ext}_R^{m_0+3}(N/J^\ell N, R) \to \mathrm{Ext}_R^{m_0+3}(N, R) = 0\ . \end{aligned}$$

From $(\star)$, we have

$$\begin{aligned} \text{length}\, \mathrm{Ext}_R^{m_0+1}(J^\ell N, R) &= \text{length}\, \mathrm{Ext}_R^{m_0+2}(N/J^\ell N, R)\ , \\ \text{length}\, \mathrm{Ext}_R^{m_0+2}(J^\ell N, R) &= \text{length}\, \mathrm{Ext}_R^{m_0+3}(N/J^\ell N, R)\ . \end{aligned}$$

Thus

$$0 = \mathrm{Ext}_R^{m_0+2}((\Omega_{m_0+3})^*, R) = \mathrm{Ext}_R^1(\Omega_2^*, R) \neq 0\ ,$$

which is a contradiction. □

7. Rainer Schultz gave the following example from which we know that **[T2] is not true for artinian rings in general** by Lemma 2.9. Thus **[NC] is not true for artinian rings in general**.
8. There is a self-injective artinian ring R and a finitely generated left R-module ${}_RM$ such that
   (i) $\mathrm{Ext}^i_R(M, M) = 0$ for any $i > 0$
   (ii) $M_{\mathrm{End}_R(M,M)}$ is not finitely generated $\mathrm{End}_R(M, M)$-module
9. Robert Martinez-Villa explored conditions in the category of functors of the stable category which are equivalent to [NC].

   Theorem 8.7 (Robert Martinez-Villa). *Assume ℓ.dom.dim $A \geq n$. Then for any $k \leq n$, $Dom_k = \{{}_AM | \ \ell.\mathrm{dom.dim}\ M \geq k\}$ is contravariantly finite in the stable category $\underline{mod}$-A of the module category.*

   We set
   $$\tilde{\mathfrak{F}}_k = \{F \in mod(\underline{mod}\text{-}A) | F(M) = 0 \text{ for any } M \in Dom_k\}$$
   $$\tilde{\mathfrak{T}}_k = \{G \in mod(\underline{mod}\text{-}A) | G(M) = 0 \text{ for any } M \in \tilde{\mathfrak{F}}_k\}$$

   Then we know $(\tilde{\mathfrak{T}}_k, \tilde{\mathfrak{F}}_k)$ is a hereditary torsion theory with a torsion radical t_k. We denote $Dom = \bigcap_{k=0}^{\infty} Dom_k$ and $\tilde{\mathfrak{F}} = \bigcap_{k=0}^{\infty} \tilde{\mathfrak{F}}_k$.
   Let $(\tilde{\mathfrak{T}}, \tilde{\mathfrak{F}})$ be a corresponding torsion theory with a torsion radical t.
   Robert Martinez-Villa gave the following conjecture.

   Conjecture 10 (MC: Martinez Conjecture [10]). *For any $M \in mod(\underline{mod}$-$A)$, it holds that*
   (1) $t(M) = \bigcap_{k=0}^{\infty} t_k(M)$
   (2) *$t(M)$ is finitely presented*

   Theorem 8.8 (Martinez-Villa Roberto [9]). [MC] *implies* [NC].
10. Cheng Chang Xi [16] showed that *dominant dimension is not invariant under derived equivalences.*

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Ram Parkash Sharma and Meenakshi

On construction of global actions for partial actions

Abstract: Every partial action α on a set has a unique minimal globalization (up to equivalence [4]). In [2], the authors proved that $\{(G, X_G), i\}$ is a unique minimal globalization of α which is equivalent to left coset action $(G, G/G_x)$ for any $x \in X$, if α is a transitive partial action. In this paper, we construct a unique minimal global action of a given partial action which is equivalent to $(G, \cup_{s\in S} G/G_s)$, where S is a G-transversal in X without resorting to transitivity so that a minimal global action can be constructed for a larger class of partial actions on sets. We also study the Hausdroff topology on a minimal globalization.

Keywords: Patial Actions; Global Actions; Actions of Topological Groups.

1 Introduction

Throughout this paper, we assume that G is a group under multiplication gh $(g, h \in G)$ with the identity denoted by e. Recall that a group action of G on a set Y is a function $G \times Y \to Y$, $(g, x) \mapsto g \cdot x$ such that $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$, $x \in Y$. A partial action α of G on a set X is a pair $\alpha = \{\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}\}$, where for each $g \in G$, D_g is a subset of X and $\alpha_g \colon D_{g^{-1}} \to D_g$ is a bijective map, satisfying the following three properties:
(i) $D_e = X$ and $\alpha_e = Id_X$, the identity map on X,
(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$,
(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for $x \in D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$, for each $g, h \in G$.

If α is a partial action of G on X, then we say that X is a partial G–set and the partial action α will be denoted by (α, X). Similarly, the group action of G on Y is denoted by (G, Y).

Following [2], for convenience we write $\exists g \cdot x$ to mean that $g \cdot x$ is defined; more precisely, $\exists g \cdot x$ means $x \in D_{g^{-1}}$ and $g \cdot x \in D_g$. Thus, we have an equivalent definition as follows:

The partial action α of G on X defines a partial function from $G \times X$ to X which satisfies the following conditions:
(PA1) $\exists e \cdot x$ for all $x \in X$ and $e \cdot x = x$;

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(PA2) $\exists g \cdot x$ implies that $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$;
(PA3) $\exists g \cdot (h \cdot x)$ implies that $\exists (gh \cdot x)$ and $g \cdot (h \cdot x) = (gh) \cdot x$.

When a partial action on some structure is given, then one of the most relevant problem is the question of the existence and uniqueness of a globalization; that is, a global group action (enveloping action) whose restriction to the original object gives the initial partial action. The definition of a global action of a partial action and the exact solution of the problem depend upon the category under consideration. For example, in the category of K–algebras and semialgebras partial action α has a unique global action if and only if each D_g is a unital algebra and unital semialgebra respectively (i.e. each D_g is generated by a central idempotent) [3, 5]; whereas in the category of the topological spaces, for each partial action there always exists a unique global action (see [4]).

On the other hand, for a given action of G on a set Y, $G \times Y \to Y, (g, x) \mapsto g \cdot x$ and a non-empty set $X \subset Y$, each element $g \in G$ induces a partial bijection of X whose domain is given by $D_{g^{-1}} = \{x \in X | g \cdot x \in X\}$ and hence there is a natural partial action α of G on X defined by $\alpha_g \colon D_{g^{-1}} \to X$, $\alpha_g(x) = g \cdot x$. In this case, the partial action arises by restricting the global action.

The following definitions are essential for the development of this paper which are taken from [2].

Definition 1.1. *Two partial actions α and α' of a group G on the sets X and X' are said to be equivalent if there exists a bijection $f \colon X \to X'$ such that for each $x \in X$, $\exists g \cdot x$ in X if and only if $\exists g \cdot f(x)$ in X' and in this case $f(g \cdot x) = g \cdot f(x)$. Such a map f is called an isomorphism between G–actions α and α'.*

Definition 1.2. *A partial action α of a group G on a set X has a globalization (G, Y) if there exists an injection $i \colon X \to Y$ such that the partial action α and the induced partial action of G on X are equivalent.*

Definition 1.3. *A globalization $\{(G, X), i\}$ of the partial action α is said to be minimal if for any globalization $\{(G, X'), i'\}$ of α, there exists an injection $\lambda \colon X \to X'$ such that $\lambda(g \cdot x) = g \cdot \lambda(x)$ for all $g \in G$ and $x \in X$.*

J. Kellendonk and M. V. Lawson in [4] proved that every partial action on a set has a unique minimal globalization (up to equivalence). In [2], K. Choi and Y. Lim proved that $\{(G, X_G), i\}$ is a unique minimal globalization of α up to equivalence, where X_G consists of the equivalence classes with the equivalence relation $\sim$ defined on $G \times X$ by $(g, x) \sim (h, y)$ if and only if $\exists (h^{-1}g) \cdot x$ and $(h^{-1}g) \cdot x = y$. The action of G on X_G is given by $(h, [g, x]) \mapsto [hg, x]$ and $i \colon X \to X_G$ is given by $x \mapsto [e, x]$. Further, they proved that for a transitive partial action α on X, the action (G, X_G) is equivalent to the left action $(G, G/G_x)$ for any $x \in X$, where $G_x = \{g \in G | \exists g \cdot x \text{ and } g \cdot x = x\}$. This helps in constructing a minimal global action of a transitive partial action. We started this paper with the aim to obtain a generalization of these results for any partial action without resorting to transitivity so that for a larger class of partial actions, a unique

minimal globalization can be constructed naturally using left coset actions. For this, we want to know the exact size of an orbit O_x in (G, X_G) for $x \in X$ in terms of the corresponding orbit in the partial action α using the elements of $G^x = \{g \in G | \exists g \cdot x\}$ (for G^x see [2]). In the process of this globalization, we also prove the orbit stabilizer theorem for partial actions; that is, for each $x \in X$, (α, O_x^α) is equivalent to $(\alpha', G^x/G_x)$ and (α, X) is equivalent to $(\alpha', \cup_{s\in S} G^s/G_s)$, where S is a partial G– transversal in X; that is, a subset of X which meets each partial orbit exactly once. Hence $|O_x^\alpha| = |G^x|/|G_x|$, if G is finite. Finally, we study partial actions of topological groups and prove that there exists a Hausdroff topology on X_G such that the action of G on X_G is continuous if and only if each G_s is closed in G for $s \in S$.

2 Global Actions of Partial Actions on Sets

Choi and Lim [2] proved that a minimal globalization of each transitive partial action is equivalent to a group coset action. In this section, we show that the minimal globalization of each partial group action α on a set X is equivalent to a group action of G on $\cup_{s\in S}(G/G_s)$, where S is a G–transversal in X. We need the following results.

Proposition 2.1. *Let (α, X) be a partial action of G. Suppose that $u, v \in X$ are in the same orbit and let G_u and G_v be their respective stabilizer subgroups. Then there exists an element $g \in G$ such that $G_u = gG_vg^{-1}$.*

Proof. Suppose (G, Y) be a minimal global action of (α, X). Then by [1], $O_x^\alpha = O_x \cap X$, where O_x^α and O_x are the orbits of x in X and Y respectively. Further the stabilizer of x in G is same in (α, X) and (G, Y), so the result follows for (α, X) as it is true for (G, Y). □

As a consequence of the above result, we have

Corollary 2.1. *Let G act on X partially and $u, v \in O_x^\alpha$ for some $x \in X$. if G_u is not normal in G, then G_v is also not normal in G.*

We also get a similar result to that of the orbits in group actions on sets.

Proposition 2.2. *Two orbits in a partial action are either identical or disjoint. Moreover, if $y_1, y_2 \in O_x^\alpha$, then $O_{y_1}^\alpha = O_{y_2}^\alpha$.*

Let X' be a subset of X. Then we say that (α, X') exists (α is a partial action of G on X') if for $y \in X'$, $\exists g^{-1} \cdot y$ implies that $g^{-1} \cdot y \in X'$. A natural example of such a subset is provided by the following result.

Lemma 2.1. *Let (α, X) be a partial action of a group G. Then for each $x \in X$, (α, O_x^α) exists.*

Proof. Let $y \in O_x^\alpha$ such that $\exists g^{-1} \cdot y$. Then $g^{-1} \cdot y \in O_x$, where O_x is the orbit of x in a minimal global action (G, Y) of (α, X), which always exists. But by ([1], Proposition 4), $O_x^\alpha = O_x \cap X$. Hence $g^{-1} \cdot y \in O_x^\alpha$. □

In order to generalize the main result of this paper from a transitive partial action to any partial action, we need a relation between orbits and stabilizers of a partial action analogous to orbit stabilizer theorem for group actions. But first we note that the orbit stabilizer theorem is not true, in general, for partial actions as observed below:

Example 1. *Let $X = \{x_1, x_2, x_3, x_4\}$ be a set and $G = \{g | g^8 = 1\}$ be a cyclic group of order 8. Consider a partial action α of G on X given by:*
$\exists e \cdot x$ for all x, $\exists g^2 \cdot x_1$, $\exists g^2 \cdot x_2$, $\exists g^2 \cdot x_3$, $\exists g^4 \cdot x_1$, $\exists g^4 \cdot x_2$, $\exists g^6 \cdot x_1$, $\exists g^6 \cdot x_2$ and $\exists g^6 \cdot x_4$ such that $e \cdot x = x\ \forall x \in X$; $g^2 \cdot x_1 = x_2$, $g^2 \cdot x_2 = x_1$, $g^2 \cdot x_3 = x_4$; $g^4 \cdot x_1 = x_1$, $g^4 \cdot x_2 = x_2$, $g^6 \cdot x_1 = x_2$, $g^6 \cdot x_2 = x_1$, $g^6 \cdot x_4 = x_3$. Therefore, we have $G_{x_1} = \{e, g^4\}$ so $[G : G_{x_1}] = 4$, but $|O_{x_1}^\alpha| = 2$ as $O_{x_1}^\alpha = \{x_1, x_2\}$. Similarly, $G_{x_3} = \{e\}$ and $O_{x_3}^\alpha = \{x_3, x_4\}$. Hence, no element of X satisfies the orbit stabilizer theorem.

Note that, if there is some $x \in X$ such that $\exists g \cdot x$ for all $g \in G$, then the orbit stabilizer theorem holds for that x, as in this case $O_x = O_x^\alpha$ and G_x is same in partial action (α, X) and minimal globalization action (G, Y). Thus, consider the subset $G^x = \{g^{-1} \in G | \exists g^{-1} \cdot x\}$ of G (see [2]) and define $G^x/G_x = \{g^{-1}G_x | \exists g^{-1} \cdot x\}$. Then a natural question arises: Does there exist any partial action γ induced by α such that (α, O_x^α) and $(\gamma, G^x/G_x)$ are equivalent?

We observe that there is a natural partial action of G on G^x/G_x which is a restriction of the left coset action $(G, G^x/G_x)$ for any $x \in X$. Further, it is also shown that G^x/G_x is partially isomorphic to O_x^α. These results are used to prove that the given partial action (α, X) is the restriction of left coset action $(G, \cup_{s \in S} G/G_s)$. First we prove that G^x/G_x is a partial G–set.

Lemma 2.2. *Let $x \in X$ and G^x/G_x be as above, then $(\alpha', G^x/G_x)$ exists where α' is a partial action of G such that for $g^{-1}G_x \in G^x/G_x$ (that is, $\exists g^{-1} \cdot x$), $\exists h \cdot (g^{-1}G_x)$ if and only if $\exists g^{-1} \cdot x$ and $\exists hg^{-1} \cdot x$. In this case $h \cdot g^{-1}G_x = hg^{-1}G_x$.*

Proof. For $x \in X$, there exists a natural group action of G on G/G_x, that is, $h \cdot (g^{-1}G_x) = hg^{-1}G_x$ for any $g, h \in G$. This left coset action of G on G/G_x induces a partial action on its subset G^x/G_x as follows: Each element $h \in G$ induces a partial bijection of G^x/G_x whose domain is given by $\{g^{-1}G_x \in G^x/G_x | hg^{-1} \cdot G_x \in G^x/G_x\}$ and there is a natural partial action α' of G (induced by α) on G^x/G_x defined by

$$\alpha'_h : \{g^{-1}G_x | hg^{-1} \cdot x \in G^x/G_x\} \to G^x/G_x,$$

$g^{-1}G_x \mapsto hg^{-1} \cdot G_x$. Note that α'_h acts on those elements $g^{-1}G_x$ of G^x/G_x for which $\exists hg^{-1} \cdot x$. Hence, the result follows from the definition of G^x/G_x. □

Now we prove

Theorem 2.1 (The Orbit Stabilizer Theorem). *Let (α, X) be a partial action of a group G on X. Then for each $x \in X$, (α, O^{α}_{x}) is equivalent to $(\alpha', G^{x}/G_{x})$ and (α, X) is equivalent to $(\alpha', \cup_{s\in S} G^{s}/G_{s})$. Hence $|O^{\alpha}_{x}| = |G^{x}|/|G_{x}|$, if G is finite.*

Proof. Define a map $f\colon O^{\alpha}_{x} \to G^{x}/G_{x}, x \in X$, by $f(g^{-1}x) = g^{-1}G_{x}$, for $\exists g^{-1}\cdot x$. Note that $\exists g^{-1}\cdot x$ implies $g^{-1}G_{x} \in G^{x}/G_{x}$. Suppose that $\exists g^{-1}\cdot x$ and $\exists h^{-1}\cdot x$ such that $g^{-1}\cdot x = h^{-1}\cdot x$. Now by using (PA2), $\exists h\cdot(h^{-1}\cdot x)$, that is, $\exists h\cdot(g^{-1}\cdot x)$ and $h\cdot(g^{-1}\cdot x) = h\cdot(h^{-1}\cdot x) = x$ which implies that $\exists h\cdot(g^{-1}G_{x})$ and $h\cdot(g^{-1}G_{x}) = hg^{-1}G_{x} = G_{x}$, so that $g^{-1}G_{x} = h^{-1}G_{x}$, proving that f is well-defined. Reversing the steps, f is one-one. Obviously, f is onto. Since $X = \cup_{s\in S} O^{\alpha}_{s}$ (disjoint union of partial orbits), it follows from the above lemma that (α, X) is equivalent to $(\alpha', \cup_{s\in S} G^{s}/G_{s})$. ☐

Example 2. *Consider the partial action α of G on X given in Example 1. Here $O^{\alpha}_{x_1} = \{x_1, x_2\}$ and $O^{\alpha}_{x_3} = \{x_3, x_4\}$. Let $S = \{x_1, x_3\}$ be a G–transversal in X. Then $G^{x_1} = \{e, g^2, g^4, g^6\}$ and $G_{x_1} = \{e, g^4\}$. Since $g^4 G_{x_1} = G_{x_1}$ and $g^6 G_{x_1} = g^2 G_{x_1}$, we have $G^{x_1}/\, G_{x_1} = \{G_{x_1}, g^2 G_{x_1}\} \simeq O^{\alpha}_{x_1}$ under the map $x_1 \mapsto G_{x_1}$ and $x_2 \mapsto g^2 G_{x_1}$.*

Similarly, $G^{x_3} = \{e, g^2\}$ and $G_{x_3} = \{e\}$. So $G^{x_3}/\, G_{x_3} = \{G_{x_3}, g^2 G_{x_3}\} \simeq O^{\alpha}_{x_3}$ under the map $x_3 \mapsto G_{x_3}$ and $x_4 \mapsto g^2 G_{x_3}$. Therefore, we have $f\colon X \to \cup_{s\in S} G^{s}/G_{s}$ with $S = \{x_1, x_3\}$, which is a bijection. Consider $\cup_{s\in S} G^{s}/G_{s} = \{G_{x_1}, g^2 G_{x_1}, G_{x_3}, g^2 G_{x_3}\}$. Using the given partial action α, we have $\exists e\cdot G_{x}, \forall x \in S, \exists g^2\cdot G_{x_1}, \exists g^4\cdot G_{x_1}, \exists g^6\cdot G_{x_1}, \exists e\cdot (g^2 G_{x_1}), \exists g^2\cdot(g^2 G_{x_1}), \exists g^4\cdot(g^2 G_{x_1}), \exists g^2\cdot G_{x_3}, \exists e\cdot(g^2 G_{x_3}), \exists g^2\cdot G_{x3}$. The action on these elements will be according to the rule given above. Now, for $g \in G$, we have $f(g^2(g^2 G_{x_1}) = f(g^4 G_{x_1}) = f(G_{x_1}) = x_1$ and $g^2 f(g^2 G_{x_1}) = g^4\cdot x_1 = x_1$. Thus, $f(g^2(g^2 G_{x_1}) = g^2\cdot f(g^2 G_{x_1})$. Similarly, it can be verified for the other elements of G, proving that (α, X) is equivalent to $(\alpha', \cup_{s\in S} G^{s}/G_{s})$.

Recall from [2] that $G^{x}/G_{x} = \{g^{-1}G_{x} | \exists g^{-1}\cdot x\}$. So $(G^{x}/G_{x})_{G}$ consists of equivalence classes $[h, g^{-1}G_{x}]$ such that $\exists g^{-1}\cdot x$ and action of G on $(G^{x}/G_{x})_{G}$ is given by $h'[h, g^{-1}G_{x}] = [h'h, g^{-1}G_{x}]$. By Lemma 2.1 of [2], $h[g^{-1}, G_{x}] = [hg^{-1}, G_{x}] = [h, g^{-1}G_{x}]$, as $\exists g^{-1}\cdot x$. Hence $[e, g^{-1}G_{x}] = [g^{-1}, G_{x}] \in (G^{x}/G_{x})_{G}$ and we have

Lemma 2.3. *Let (α, X) be a partial action of a group G on X and $x \in X$. Then $(G, (G^{x}/G_{x})_{G})$ is a minimal globalization of $(\alpha', G^{x}/G_{x})$.*

Proof. Since α' is transitive on (G^{x}/G_{x}), the result follows from ([2], Theorem 2.2). ☐

Lemma 2.4. *Let (α, X) be a partial action of a group G on X. Then $(G, (\cup_{s\in S} G^{s}/G_{s})_{G})$ is a minimal globalization of (α, X).*

Proof. Note that for any (α, X_i) and (α, X_j), if $X_i \cap X_j = \emptyset$ then $(X_i)_G \cap (X_j)_G = \emptyset$, because for $[g, x] \in (X_i)_G$ and $[h, y] \in (X_j)_G$ such that $[g, x] = [h, y]$, that is, $h^{-1}g\cdot x = y$, we have the contradiction, i.e., $X_i \cap X_j \neq \emptyset$. Therefore the disjointness of G^{s_1}/G_{s_1} and G^{s_2}/G_{s_2} for $s_1 \neq s_2$ gives that $(G^{s_1}/G_{s_1})_G$ and $(G^{s_2}/G_{s_2})_G$ are also disjoint from which it easily follows that $(G, (\cup_{s\in S} G^{s}/G_{s})_{G})$ is equivalent to $(G, \cup_{s\in S}(G^{s}/G_{s})_{G})$. Now define a map $i\colon \cup_{s\in S} G^{s}/G_{s} \to \cup_{s\in S}(G^{s}/G_{s})_{G}, g^{-1}G_{s} \to [e, g^{-1}G_{s}]$, where $\exists g^{-1}\cdot s$.

In view of the orbit stabilizer theorem, it suffices to show that the partial action induced by restricting the action $(G, \cup_{s\in S}(G^s/G_s)_G)$ on $(G, i(X))$ is equivalent to (α, X). For any $h \in G$ and $[e, g^{-1}G_s] \in \cup_{s\in S}(G^s/G_s)_G$, we have $\exists h\cdot[e, g^{-1}G_s]$ in $i(\cup_{s\in S}(G^s/G_s))$ if and only if $\exists h\cdot[e, g^{-1}G_s] \in i(\cup_{s\in S}(G^s/G_s))$if and only if $[e, hg^{-1}G_s] \in i(\cup_{s\in S}(G^s/G_s))$ if and only if $\exists hg^{-1}\cdot s$. Hence the result follows from Lemma 2.2. □

Theorem 2.2. *Let (α, X) be a partial action of a group G on X. Then (G, X_G), the minimal globalization of (α, X), is equivalent to $(G, \cup_{s\in S}G/G_s)$, where S is a G–transversal in X.*

Proof. Since α' is transitive on G^x/G_x and $(G, (G^x/G_x)_G)$ is the minimal global action of $(\alpha', G^x/G_x)$ (unique up to equivalence), therefore by ([2],Theorem 2.6), we have $(G, (G^x/G_x)_G)$ is equivalent to left coset action $(G, G/G_x)$ for any G^x/G_x. As for $s_1 \neq s_2 \in S$, $G/G_{s_1}, G/G_{s_2}$ are disjoint. Thus, $(G, (\cup_{s\in S}G^s/G_s)_G)$ is equivalent to $(G, \cup_{s\in S}G/G_s)$. By the orbit-stabilizer theorem, it follows that (G, X_G) is equivalent to $(G, \cup_{s\in S}G/G_s)$. □

As an application of the orbit stabilizer theorem for partial actions, we have the following result which is useful in construction of a minimal global action.

Corollary 2.2. *Let G be a finite group and (G, Y) be a minimal global action of (α, X). Then for each $x \in X$,*

$$|O_x| = |O_x^\alpha| + (|\overline{G^x}|)/(|G_x|),$$

where $\overline{G^x} = \{g \in G | g \notin G^x\}$.

Proof. By the orbit stabilizer theorem for partial actions, we have

$$|O_x^\alpha| = |G^x|/|G_x|.$$

Since $|G_x| \mid |G|$ and $|G_x| \mid |G^x|$. Therefore $|G_x| \mid |G| - |G^x|$, that is, $|G_x| \mid |\overline{G^x}|$, as $|\overline{G^x}| = |G| - |G^x|$. So using the orbit -stabilizer theorem for global actions, we get

$$\begin{aligned}|O_x| &= |G|/|G_x|\\ &= (|G^x| + |\overline{G^x}|)/|G_x|\\ &= |G^x|/|G_x| + |\overline{G^x}|/|G_x|\\ &= |O_x^\alpha| + |\overline{G^x}|/|G_x|\,.\end{aligned}$$ □

The above corollary together with Theorem 2.2 discribes the minimal global action of a partial action.

Example 3. *Consider the partial action given in Example 1. We have $|G_{x_1}| = 2$, $|O_{x_1}^\alpha| = 2$, $|G^{x_1}| = 4$ and hence $|\overline{G^{x_1}}| = 4$. Therefore, $|O_{x_1}| = |O_{x_1}^\alpha| + |\overline{G^{x_1}}|/|G_{x_1}|$ yields $|O_{x_1}| = 2 + 2 = 4$. Similarly for x_3, $|O_{x_3}^\alpha| = 2$, $|G^{x_3}| = 2$, $|G_{x_3}| = 1$ and $|\overline{G^{x_3}}| = 6$. Therefore, we have $|O_{x_3}| = 2 + 6 = 8$. Taking $S = \{x_1, x_3\}$, we have $|O_{x_1}| = 4$ and $|O_{x_3}| = 8$. So $G/G_{x_1} = \{G_{x_1}, gG_{x_1}, g^2G_{x_1}, g^3G_{x_1}\}$ and $G/G_{x_3} = \{G_{x_3}, gG_{x_3}, g^2G_{x_3}, g^3G_{x_3}, g^4G_{x_3}, g^5G_{x_3}, g^6G_{x_3},$*

$g^7G_{x_3}\}$. Since $G^{x_1}/G_{x_1} \simeq O^{\alpha}_{x_1}$ under the map $G_{x_1} \mapsto x_1, gG_{x_1} \mapsto x_2$ and $G^{x_3}/G_{x_3} \simeq O^{\alpha}_{x_3}$ under the map $G_{x_3} \mapsto x_3, g^2G_{x_3} \mapsto x_4$, we can identify the elements of G/G_{x_1} by x_1, x_5, x_2, x_6 respectively and the elements of G/G_{x_3} by $x_3, x_7, x_4, x_8, x_9, x_{10}, x_{11}, x_{12}$ respectively and thus the coset action $(G, \cup_{s\in S}G/G_s)$ gives the table.

β_g	β_{g^2}	β_{g^3}	β_{g^4}	β_{g^5}	β_{g^6}	β_{g^7}	β_1
$x_1 \longmapsto x_5$	$x_1 \longmapsto x_2$	$x_1 \longmapsto x_6$	$x_1 \longmapsto x_1$	$x_1 \longmapsto x_5$	$x_1 \longmapsto x_2$	$x_1 \longmapsto x_6$	$x_1 \longmapsto x_1$
$x_2 \longmapsto x_6$	$x_2 \longmapsto x_1$	$x_2 \longmapsto x_5$	$x_2 \longmapsto x_2$	$x_2 \longmapsto x_6$	$x_2 \longmapsto x_1$	$x_2 \longmapsto x_5$	$x_2 \longmapsto x_2$
$x_3 \longmapsto x_7$	$x_3 \longmapsto x_4$	$x_3 \longmapsto x_8$	$x_3 \longmapsto x_9$	$x_3 \longmapsto x_{10}$	$x_3 \longmapsto x_{11}$	$x_3 \longmapsto x_{12}$	$x_3 \longmapsto x_3$
$x_4 \longmapsto x_8$	$x_4 \longmapsto x_9$	$x_4 \longmapsto x_{10}$	$x_4 \longmapsto x_{11}$	$x_4 \longmapsto x_{12}$	$x_4 \longmapsto x_3$	$x_4 \longmapsto x_7$	$x_4 \longmapsto x_4$
$x_5 \longmapsto x_2$	$x_5 \longmapsto x_6$	$x_5 \longmapsto x_1$	$x_5 \longmapsto x_5$	$x_5 \longmapsto x_2$	$x_5 \longmapsto x_6$	$x_5 \longmapsto x_1$	$x_5 \longmapsto x_5$
$x_6 \longmapsto x_1$	$x_6 \longmapsto x_5$	$x_6 \longmapsto x_2$	$x_6 \longmapsto x_6$	$x_6 \longmapsto x_1$	$x_6 \longmapsto x_5$	$x_6 \longmapsto x_2$	$x_6 \longmapsto x_6$
$x_7 \longmapsto x_4$	$x_7 \longmapsto x_8$	$x_7 \longmapsto x_9$	$x_7 \longmapsto x_{10}$	$x_7 \longmapsto x_{11}$	$x_7 \longmapsto x_{12}$	$x_7 \longmapsto x_3$	$x_7 \longmapsto x_7$
$x_8 \longmapsto x_9$	$x_8 \longmapsto x_{10}$	$x_8 \longmapsto x_{11}$	$x_8 \longmapsto x_{12}$	$x_8 \longmapsto x_3$	$x_8 \longmapsto x_7$	$x_8 \longmapsto x_4$	$x_8 \longmapsto x_8$
$x_9 \longmapsto x_{10}$	$x_9 \longmapsto x_{11}$	$x_9 \longmapsto x_{12}$	$x_9 \longmapsto x_3$	$x_9 \longmapsto x_7$	$x_9 \longmapsto x_4$	$x_9 \longmapsto x_8$	$x_9 \longmapsto x_9$
$x_{10} \longmapsto x_{11}$	$x_{10} \longmapsto x_{12}$	$x_{10} \longmapsto x_3$	$x_{10} \longmapsto x_7$	$x_{10} \longmapsto x_4$	$x_{10} \longmapsto x_8$	$x_{10} \longmapsto x_9$	$x_{10} \longmapsto x_{10}$
$x_{11} \longmapsto x_{12}$	$x_{11} \longmapsto x_3$	$x_{11} \longmapsto x_7$	$x_{11} \longmapsto x_4$	$x_{11} \longmapsto x_8$	$x_{11} \longmapsto x_9$	$x_{11} \longmapsto x_{10}$	$x_{11} \longmapsto x_{11}$
$x_{12} \longmapsto x_3$	$x_{12} \longmapsto x_7$	$x_{12} \longmapsto x_4$	$x_{12} \longmapsto x_8$	$x_{12} \longmapsto x_9$	$x_{12} \longmapsto x_{10}$	$x_{12} \longmapsto x_{11}$	$x_{12} \longmapsto x_{12}$

The global action given in above table is the unique minimal global action (up to equivalence) of the partial action given in Example 1.

3 Partial actions of topological groups

If $X_1, X_2, \ldots$ are topological spaces with topologies $\tau_1, \tau_2, \ldots$ respectively. Then obviously, $\cup\tau_i$ may not be a topology on $\cup X_i$. However, we can define a topology τ on $X = \cup X_i$ through the canonical injections $\phi_i : X_i \to X$. A subset U of X is said to be open in X if $\phi_i^{-1}(U)$ is open in X_i for each i. Since $\phi_j^{-1}(\cup U_i) = \cup\phi_j^{-1}(U_i)$ and $\phi_j^{-1}(\cap_{i=1,2\ldots n}U_i) = \cap_{i=1,2\ldots n}\phi_j^{-1}(U_i)$ for each j; τ is a topology on X. If $X_i \cap X_j$ is non empty for some $i \neq j$, then X_i or X_j may not belong to τ. For example, if $X = \{a, b, c, d\} = X_1 \cup X_2$, where $X_1 = \{a, b, c\}$ and $X_2 = \{c, d\}$ with $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X_1\}$ and $\tau_2 = \{\emptyset, \{c\}, X_2\}$, then by definition, the topology τ on X is given by $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$.

Here $X_2 \notin \tau$ as $\phi_1^{-1}(X_2) = \{c\} \notin \tau_1$. However, we have

Lemma 3.1. *If all X_i are mutually disjoint, then each X_i is in τ.*

Proof. This follows because $\phi_i^{-1}(X_i) = X_i$ and $\phi_j^{-1}(X_i) = \emptyset$ for $i \neq j$. □

Remark 3.1. *If $x, y \in X_i$ are separated by two disjoint open sets in τ_i, then x, y may not be separated by two disjoint open sets in τ. For example, if $X = \{a, b, c, d\} = X_1 \cup X_2$, where $X_1 = \{a, b, c\}$ and $X_2 = \{c, d\}$ with $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X_1\}$ and $\tau_2 = \{\emptyset, \{c\}, \{d\}, X_2\}$; then by definition, the topology τ on X is given by: $\tau = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}, X\}$. Here c and d are separated by open sets $\{c\}$ and $\{d\}$ in τ_2, but there are no disjoint open sets containing these elements in τ.*

However, we have

Lemma 3.2. *If all X_i are mutually disjoint, then X_i is a Hausdroff space for each i if and only if $X = \cup_{i\in J} X_i$ is a Hausdroff space.*

Proof. Suppose that each $X_i(i \in J$(indexing set)) is a Hausdroff space and $x, y \in X$ ($x \neq y$) are in X_i and X_j respectively, where $i \neq j$, then by Lemma 3.1, x and y are separated by two disjoint open sets X_i and X_j in τ. If $x, y \in X_i$ for some i, then $\exists U_x, U_y \in \tau_i$ containing x and y respectively such that $U_x \cap U_y = \emptyset$. Since $X_i^{\prime}s$ are mutually disjoint, U_x and U_y are also open sets in τ.

Conversely, let $\cup_{i\in J} X_i$ (disjoint union) be a Hausdroff space. Then for any distinct elements $x, y \in X_i \subseteq \cup_{i\in J} X_i$, there exist two open sets U_x and U_y in τ containing x and y respectively such that $U_x \cap U_y = \emptyset$. Then $\phi_i^{-1}(U_x) = U_x \cap X_i$ and $\phi_i^{-1}(U_y) = X_i \cap U_y$ are open sets in τ_i containing x and y respectively such that $(U_x \cap X_i) \cap (X_i \cap U_y) = \emptyset$. □

Now for the remaining of the section we assume that G is a topological group. Then the subset $G^x$of G is a topological space having subspace topology and hence G^x/G_x is a topological space having quotient topology on it. By the previous section, we have (α, X) is equivalent to $(\alpha', \cup_{s\in S} G^s/G_s)$ where S is a G–transversal. Thus X become a topological space with respect to topology τ defined above, that is, any set U is in τ if $\phi_s^{-1}(U)$ is open set of G^s/G_s for each $s \in S$.

Proposition 3.1. *The following statements are equivalent.*
(i) There exists a Hausdroff topology on X_G such that the group action of G on X_G is continuous.
(ii) G_s is closed for each $s \in S$.
(iii) G_x is closed for all $x \in X$.

Proof. Since G is a topological group, each translation by an element of G is a homeomorphism. Then by ([2], Proposition 4.1), we have (ii) and (iii) are equivalent.
$(i) \Rightarrow (ii)$: Suppose that there exists a Hausdroff topology on X_G such that the group action of G on X_G is continuous, that is, $\cup_{s\in S} G/G_s$ is a Hausdroff topological space and the action of G on $\cup_{s\in S} G/G_s$ is continuous. Since this union is mutually disjoint, by Lemma 3.2, each G/G_s is a Hausdroff space for each $s \in S$. Therefore, G_s must be closed for each $s \in S$.
$(ii) \Rightarrow (i)$: Suppose that G_s is closed for each $s \in S$, then the coset space G/G_s is a Hausdroff space and the disjoint union is also a Hausdroff space. So it only remains to show that the action of G on $\cup_{s\in S} G/G_s$ is continuous. For, define a map $\varphi_s \colon G \times G/G_s \to G/G_s, (g, hG_s) \mapsto ghG_s$. Then each φ_s is continuous (see [2], Proposition 4.1), that is, if U_s is a open set in G/G_s, then $\varphi_s^{-1}(U_s)$ is also a open set in $G \times G/G_s$

Now, define a map $\varphi \colon G \times \cup_{s\in S} G/G_s \to \cup_{s\in S} G/G_s$ by $\varphi(g, hG_s) = \varphi_s(g, hG_s)$. Let U be any open set in $\cup_{s\in S} G/G_s$. Since $\varphi^{-1}(U) = \cup_{s\in S} \varphi_s^{-1}(U)$, therefore, $\varphi^{-1}(U)$ is a open set in $G \times \cup_{s\in S} G/G_s$. □

Further we have

Theorem 3.1. *Suppose that there exists $s \in S$ such that G_s is a closed subgroup of G. Then*
(a) $i(X)$ is open (respectively, dense) in X_G if and only if G^s is open (respectively, dense) in G.
(b) If the set X has a topology and G^s is open in G then the inclusion $i\colon X \to X_G$ is an open mapping if and only if the evaluation mapping $ev_s\colon G^s \ni g \mapsto g \cdot s \in X$ is continuous.

Proof. Consider the following maps: $\pi\colon G \to G/G_s$, $\varphi_s\colon G/G_s \to \cup_{s\in S} G/G_s$, $\phi\colon \cup_{s\in S} G/G_s \to X_G$ and $i\colon X \hookrightarrow X_G$. Then, we have $\pi^{-1}(\varphi_s^{-1}(\phi^{-1}(i(X)))) = G^s$.
(a) Since π is a quotient mapping, G^s is open if and only if $\varphi_s^{-1}(\phi^{-1}(i(X)))$ is open in $\cup_{s\in S} G/G_s$ if and only if $\phi^{-1}(i(X))$ is open in G/G_s if and only if $i(X)$ is open in X_G. Since π is an open and continuous onto mapping, the set G^s is dense in G if and only if $\varphi_s^{-1}(\phi^{-1}(i(X)))$ is dense in $\cup_{s\in S} G/G_s$ if and only if $\phi^{-1}(i(X))$ is dense in G/G_s if and only if $i(X)$ is dense in X_G.
(b) Suppose that X is a topological space and G^s is open in G. For any open subset U of X, we have $\pi^{-1}(\varphi_s^{-1}(\phi^{-1}(i(U)))) = ev_s^{-1}(U)$. Therefore, since G^s is open in G, the set $ev_s^{-1}(U)$ is open in G^s if and only if $ev_s^{-1}(U)$ is open in G.

Thus by using (a), we have the injection $i\colon X \to X_G$ is an open mapping if and only if $ev_s^{-1}(U)$ is open for an open set U in X, that is, ev_s is continuous. □

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Meenakshi Pralhadrao Wasadikar and Karuna Tulshiram Gaikwad

On 2-absorbing and Weakly 2-absorbing Ideals in Product Lattices

Abstract: In this paper, we study some properties of 2-absorbing and weakly 2-absorbing ideals in the direct product of lattices. We show that for proper ideals I_1 and I_2 of L_1 and L_2 respectively, if $I = I_1 \times I_2$ is a 2-absorbing ideal of $L = L_1 \times L_2$, then I_1 and I_2 are 2-absorbing ideals of L_1 and L_2 respectively. We prove this property for weakly 2-absorbing ideals also. We give some characterizations of 2-absorbing and weakly 2-absorbing ideals in the direct product of lattices.

Keywords: Ideal; prime ideal; weakly prime ideal; 2-absorbing ideal; weakly 2-absorbing ideal; product lattice.

1 Introduction

Anderson and Smith [1], defined a *weakly prime ideal* in a commutative ring, as a proper ideal I of R with the property that, if $0 \neq ab \in I$ for $a, b \in R$ then either $a \in I$ or $b \in I$.

Badawi [2] defined a proper ideal I of a commutative ring R to be a *2-absorbing ideal*, if $abc \in I$ for $a, b, c \in R$ then either $ab \in I$ or $ac \in I$ or $bc \in I$.

Badawi and Darani [3] defined a proper ideal I of a commutative ring R to be a *weakly 2-absorbing ideal*, if $0 \neq abc \in I$ for $a, b, c \in R$ then either $ab \in I$ or $ac \in I$ or $bc \in I$.

Wasadikar and Gaikwad [5] introduced a 2-absorbing and a weakly 2-absorbing ideal in a lattice. A proper ideal I of a lattice L is called a *2-absorbing ideal*, if $a \wedge b \wedge c \in I$ for $a, b, c \in L$ then either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.

A proper ideal I of a lattice L with zero is called a *weakly 2-absorbing ideal*, if $0 \neq a \wedge b \wedge c \in I$ for $a, b, c \in L$ then either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.

In this paper we study 2-absorbing and weakly 2-absorbing ideals in the direct product of lattices.

In the second section we give some basic definitions and results from lattice theory. The third section deals with 2-absorbing and weakly 2-absorbing ideals in the direct product of lattices. We assume throughout that all the lattices are lattices with zero.

The undefined terms are from Grätzer [4].

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2 Preliminaries

Definition 2.1. *Let L be a lattice with zero. A proper ideal I of L is called weakly prime if $0 \neq a \wedge b \in I$ for $a, b \in L$ then either $a \in I$ or $b \in I$.*

Example 1. *Consider the lattice shown in Figure 1. The ideal $I = (f]$ is a weakly prime ideal.*

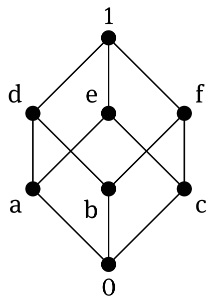

Figure 1

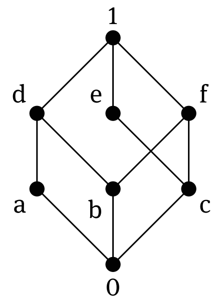

Figure 2

Every prime ideal in a lattice with zero is weakly prime. However, the converse need not hold.

Example 2. *Consider the ideal $I = (a]$ of the lattice shown in Figure 2. I is a weakly prime ideal. However, $d \wedge e = 0 \in I$, but neither $d \in I$ nor $e \in I$.*

Definition 2.2. *Let L be a lattice. A proper ideal I of L is called a 2-absorbing ideal if $a \wedge b \wedge c \in I$ for $a, b, c \in L$ then either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.*

Example 3. *Consider the lattice shown in Figure 3. The ideal $I = (f]$ is 2-absorbing.*

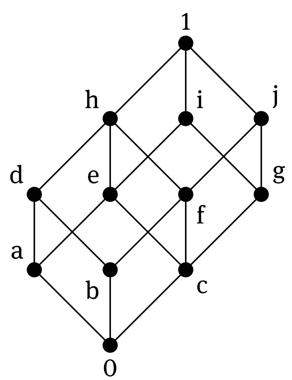

Figure 3

Definition 2.3. *Let L be a lattice with zero. A proper ideal I of L is called a weakly 2-absorbing ideal if* $0 \neq a \wedge b \wedge c \in I$ *for* $a, b, c \in L$ *then either* $a \wedge b \in I$ *or* $a \wedge c \in I$ *or* $b \wedge c \in I$.

Example 4. *Consider the lattice shown in Figure 3. The ideal* $I = (e]$ *is weakly 2-absorbing.*

Every 2-absorbing ideal in a lattice with zero is weakly 2-absorbing. However, the converse need not hold.

Example 5. *Consider the lattice shown in Figure 4. The ideal* $I = (b]$ *is weakly 2-absorbing but it is not 2-absorbing since* $i \wedge j \wedge l = 0 \in I$ *but neither* $i \wedge j = a \in I$ *nor* $i \wedge l = e \in I$ *nor* $j \wedge l = d \in I$

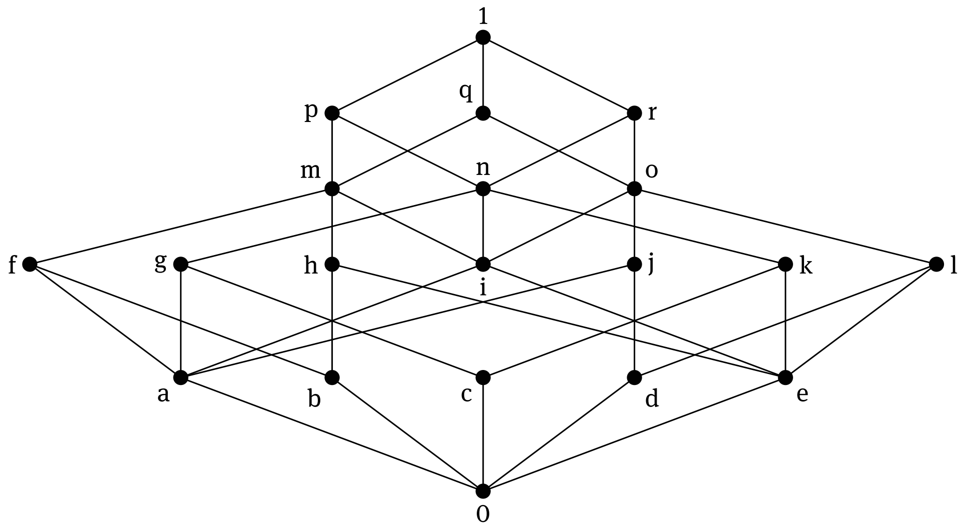

Figure 4

Remark 2.1. *If the width of a lattice L is less than or equal to two then every ideal of L is 2-absorbing.*

3 Results on Direct Product of Lattices

In this section we prove some results on 2-absorbing and weakly 2-absorbing ideals in lattices. The notion of the product lattice is from Grätzer [4, p. 27]

Theorem 3.1. *Let* $L = L_1 \times L_2$, *where* L_1 *and* L_2 *are lattices. Let* I_1 *and* I_2 *be proper ideals of* L_1 *and* L_2 *respectively. Then the following statements hold:*

1. *If $I = I_1 \times I_2$ is a 2-absorbing ideal of L then I_1 and I_2 are 2-absorbing ideals of L_1 and L_2 respectively.*
2. *If $I = I_1 \times I_2$ is a weakly 2-absorbing ideal of L then I_1 and I_2 are weakly 2-absorbing ideals of L_1 and L_2 respectively.*

Proof. (1) Suppose that $I = I_1 \times I_2$ is a 2-absorbing ideal of L. Let $a \wedge b \wedge c \in I_1$ for some $a, b, c \in L_1$. Then $(a \wedge b \wedge c, 0) \in I_1 \times I_2$. As $I_1 \times I_2$ is a 2-absorbing ideal, we have either $(a \wedge b, 0) \in I_1 \times I_2$ or $(a \wedge c, 0) \in I_1 \times I_2$ or $(b \wedge c, 0) \in I_1 \times I_2$. Hence either $a \wedge b \in I_1$ or $a \wedge c \in I_1$ or $b \wedge c \in I_1$. Thus I_1 is a 2-absorbing ideal of L_1.

Similarly, I_2 is a 2-absorbing ideal of L_2.

(2) The proof is similar as in (1). □

The following example shows that the converse of the two statements of the Theorem 3.1 need not hold.

Example 6. *Consider the lattices L_1, L_2 and $L = L_1 \times L_2$ as shown in Figure 5. Consider the ideals $I_1 = (b], I_2 = (0]$ of the lattices L_1 and L_2 respectively. Thus $I_1 \times I_2 = ((b, 0)]$. The ideals I_1 and I_2 are 2-absorbing and weakly 2-absorbing ideals of L_1 and L_2 respectively. However, $(1, 0) \wedge (c, g) \wedge (d, g) = (b, 0) \in I_1 \times I_2$, but neither $(1, 0) \wedge (c, g) = (c, 0) \in I_1 \times I_2$ nor $(1, 0) \wedge (d, g) = (d, 0) \in I_1 \times I_2$ nor $(c, g) \wedge (d, g) = (b, g) \in I_1 \times I_2$. Thus $I_1 \times I_2$ is not 2-absorbing and weakly 2-absorbing.*

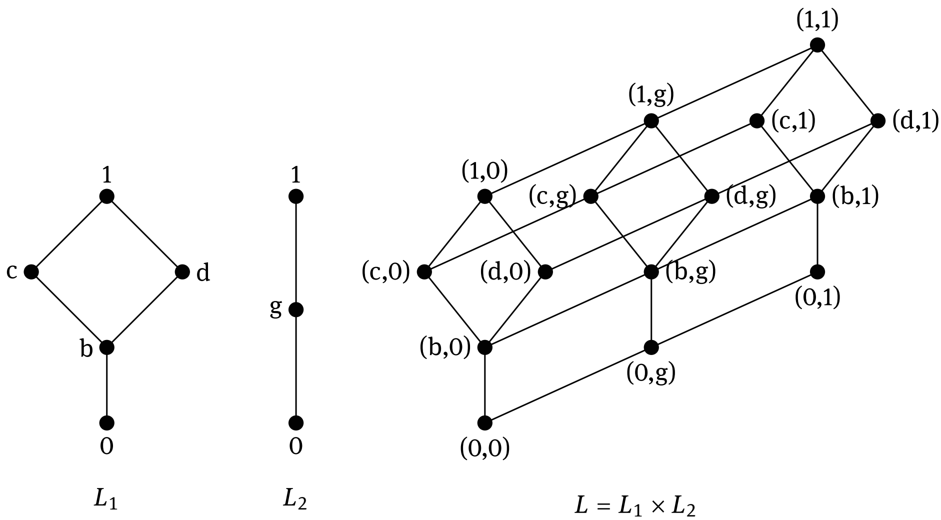

Figure 5

The following Theorem is the generalizations of the Theorem 3.1.

Theorem 3.2. *Let $L = L_1 \times L_2 \times \ldots \times L_n$, where $L_1, L_2, \ldots, L_n$ are lattices. Let $I_1, I_2, \ldots, I_n$ be proper ideals of $L_1, L_2, \ldots, L_n$ respectively. Then the following statements hold:*

1. *If $I = I_1 \times I_2 \times \ldots \times I_n$ is a 2-absorbing ideal of L then $I_1, I_2 \ldots, I_n$ are 2-absorbing ideals of $L_1, L_2 \ldots, L_n$ respectively.*
2. *If $I = I_1 \times I_2 \times \ldots \times I_n$ is a weakly 2-absorbing ideal of L then $I_1, I_2 \ldots, I_n$ are weakly -absorbing ideals of $L_1, L_2 \ldots, L_n$ respectively.*

Proof. (1) Without loss of generality, we show that I_j is a 2-absorbing ideal of L_j, where $j = 1, 2, 3, \ldots, n$. Let $a \wedge b \wedge c \in I_j$, for $a, b, c \in L_j$. Thus,
$(0, 0, \ldots, 0, a \wedge b \wedge c, 0, \ldots, 0) \in I_1 \times I_2 \times \ldots \times I_{j-1} \times I_j \times I_{j+1} \times \ldots \times I_n$.
As $I_1 \times I_2 \times \ldots \times I_{j-1} \times I_j \times I_{j+1} \times \ldots \times I_n$ is a 2-absorbing ideal, we have, either
$(0, 0, \ldots, 0, a \wedge b, 0, \ldots, 0) \in I_1 \times I_2 \times \ldots \times I_{j-1} \times I_j \times I_{j+1} \times \ldots \times I_n$ or
$(0, 0, \ldots, 0, a \wedge c, 0, \ldots, 0) \in I_1 \times I_2 \times \ldots \times I_{j-1} \times I_j \times I_{j+1} \times \ldots \times I_n$ or
$(0, 0, \ldots, 0, b \wedge c, 0, \ldots, 0) \in I_1 \times I_2 \times \ldots \times I_{j-1} \times I_j \times I_{j+1} \times \ldots \times I_n$. Hence either $a \wedge b \in I_j$ or $a \wedge c \in I_j$ or $b \wedge c \in I_j$.
(2) The proof is similar as in (1). □

Definition 3.1. *Let L be a lattice with 0 and $a, b \in L$. If $a \wedge b = 0$ implies that either $a = 0$ or $b = 0$, then L is called an Integral lattice.*

Every 2-absorbing ideal is weakly 2-absorbing. Now, in the next result we show that these are equivalent in the product of two lattices if one of the two lattices is an integral lattice.

Theorem 3.3. *Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices with zero. Let L_1 be an integral lattice and let I be a proper ideal of L_1. The following statements are equivalent:*

1. *$I \times L_2$ is a weakly 2-absorbing ideal of L.*
2. *$I \times L_2$ is a 2-absorbing ideal of L.*
3. *I is a 2-absorbing ideal of L_1.*

Proof. (1) $\Longrightarrow$ (2). Suppose that $I \times L_2$ is a weakly 2-absorbing ideal of L.
Case 1: Let $0 \neq (a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I \times L_2$ for $a_i \in L_1, b_i \in L_2$ $(i = 1, 2, 3)$. Then clearly either $(a_1, b_1) \wedge (a_2, b_2) \in I \times L_2$ or $(a_1, b_1) \wedge (a_3, b_3) \in I \times L_2$ or $(a_2, b_2) \wedge (a_3, b_3) \in I \times L_2$.
Case 2: Let $0 = (a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I \times L_2$ for $a_i \in L_1, b_i \in L_2$ $(i = 1, 2, 3)$. This implies that $a_1 \wedge a_2 \wedge a_3 = 0$. As L_1 is an integral lattice, $a_i = 0$ for some i. Without loss of generality, we may assume $a_1 = 0$. Then $(a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) = (0, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I \times L_2$. Hence $(0, b_1) \wedge (a_2, b_2) = (0, b_1 \wedge b_2) \in I \times L_2$ and $(0, b_1) \wedge (a_3, b_3) = (0, b_1 \wedge b_3) \in I \times L_2$. Thus $I \times L_2$ is a 2-absorbing ideal of L.
(2)$\Longrightarrow$ (3). Follows from Theorem 3.1 (1).
(3) $\Longrightarrow$ (1). Suppose that I is a 2-absorbing ideal of L_1. We show that $I \times L_2$ is a 2-absorbing ideal of L. Let $(a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I \times L_2$. Then $a_1 \wedge a_2 \wedge a_3 \in I$. As I is a 2-absorbing ideal of L_1, either $a_1 \wedge a_2 \in I$ or $a_1 \wedge a_3 \in I$ or $a_2 \wedge a_3 \in I$. Without loss of generality, suppose that $a_1 \wedge a_2 \in I$. Then $(a_1, b_1) \wedge (a_2, b_2) \in I \times L_2$. Hence

$I \times L_2$ is a 2-absorbing ideal of L. As every 2-absorbing ideal of L is weakly 2-absorbing ideal of L, we have $I \times L_2$ is a weakly 2-absorbing ideal of L. □

Theorem 3.4. *Let $L = L_1 \times L_2$, where L_1 and L_2 are bounded lattices and L_1 is an integral lattice. Let I be a nonzero proper ideal of L_1 and J be a nonzero ideal of L_2. The following statements are equivalent:*
1. *$I \times J$ is a weakly 2-absorbing ideal of L.*
2. *$J = L_2$ and I is a 2-absorbing ideal of L_1 or I is a prime ideal of L_1 and J is a prime ideal of L_2.*
3. *$I \times J$ is a 2-absorbing ideal of L.*

Proof. (1) $\Longrightarrow$ (2). Suppose that $I \times J$ is a weakly 2-absorbing ideal of L. If $J = L_2$ then $I \times J$ becomes $I \times L_2$. As $I \times J$ is a weakly 2-absorbing ideal of L we have $I \times L_2$ is a weakly 2-absorbing ideal of L and hence I is a 2-absorbing ideal of L_1, by Theorem 3.3. Suppose that $J \neq L_2$. We show that J is a prime ideal of L_2 and I is a prime ideal of L_1. Let $a, b \in L_2$ such that $a \wedge b \in J$, and let $0 \neq i \in I$. Then $(i, 1) \wedge (1, a) \wedge (1, b) = (i, a \wedge b) \in I \times J - \{(0, 0)\}$. As I is a proper ideal of L_1, $(1, a) \wedge (1, b) = (1, a \wedge b) \notin I \times J$. Hence either $(i, 1) \wedge (1, a) = (i, a) \in I \times J$ or $(i, 1) \wedge (1, b) = (i, b) \in I \times J$. Thus either $a \in J$ or $b \in J$. Hence J is prime ideal of L_2. Similarly, I is a prime ideal of L_1.
(2) $\Longrightarrow$ (3). If $J = L_2$ and I is a 2-absorbing ideal of L_1 then $I \times J = I \times L_2$. Then $I \times L_2$ is a 2-absorbing ideal of L by Theorem 3.3.

Suppose that I is a prime ideal of L_1 and J is a prime ideal of L_2. We show that $I \times J$ is a 2-absorbing ideal of L. Suppose that $(a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I \times J$ for some $a_1, a_2, a_3 \in L_1$ and for some $b_1, b_2, b_3 \in L_2$. As I is a prime ideal of L_1 then at least one of the $a_i^{\prime}s$ is in I, say a_1 and as J is a prime ideal of L_2 then at least one of the $b_i^{\prime}s$ is in J, say b_2 $(i = 1, 2, 3)$. Therefore $a_1 \wedge a_2 \in I$ and $b_1 \wedge b_2 \in J$, implies $(a_1 \wedge a_2, b_1 \wedge b_2) \in I \times J$. Thus $(a_1, b_1) \wedge (a_2, b_2) \in I \times J$. Hence $I \times J$ is a 2-absorbing ideal of L.
(3) $\Longrightarrow$ (1). Obvious as every 2-absorbing ideal is weakly 2-absorbing. □

Theorem 3.5. *Let $L = L_1 \times L_2$ be a bounded lattice. Let I be a nonzero proper ideal of L_1 and J be an ideal of L_2. The following statements are equivalent:*
1. *$I \times J$ is a weakly 2-absorbing ideal of L, that is not a 2-absorbing ideal.*
2. *I is a weakly prime ideal of L_1, that is not a prime ideal and $J = \{0\}$ is a prime ideal of L_2, that is L_2 is an integral lattice.*

Proof. (1) $\Longrightarrow$ (2). Suppose that $I \times J$ is a weakly 2-absorbing ideal of L, that is not a 2-absorbing ideal. Now we have to show that:
(i) I is a weakly prime ideal of L_1;
(ii) I is not a prime ideal of L_1;
(iii) $J = \{0\}$;
(iv) J is a prime ideal of L_2.

We first show (iii) and (iv).
(iii) Suppose that $J \neq \{0\}$. Then by Theorem 3.4, $I \times J$ is a 2-absorbing ideal of L, which contradicts to our hypothesis. Thus $J = \{0\}$.
(iv) Now we show that $J = \{0\}$ is a prime ideal of L_2 (and hence L_2 is an integral lattice). Suppose that for some $a, b \in L_2$, $a \wedge b \in J = \{0\}$. As I is a nonzero proper ideal of L_1, let $0 \neq i \in I$. Since $(i, 1) \wedge (1, a) \wedge (1, b) = (i, a \wedge b) \in I \times J - \{(0, 0)\}$ and as I is proper ideal of L_1, $(1, a) \wedge (1, b) = (1, a \wedge b) \notin I \times J$ we have either $(i, 1) \wedge (1, a) = (i, a) \in I \times J$ or $(i, 1) \wedge (1, b) = (i, b) \in I \times J$. Hence either $a \in J$ or $b \in J$. Thus $J = \{0\}$ is a prime ideal of L_2 and L_2 is an integral lattice.
Now we show (i) and (ii).
(i) We show that I is a weakly prime ideal of L_1. Suppose that for some $a, b \in L_1$, $a \wedge b \in I - \{0\}$. Then $(a, 1) \wedge (b, 1) \wedge (1, 0) = (a \wedge b, 0) \in I \times \{0\} - \{(0, 0)\}$ and $(a, 1) \wedge (b, 1) = (a \wedge b, 1) \notin I \times \{0\}$. We have either $(a, 1) \wedge (1, 0) = (a, 0) \in I \times \{0\}$ or $(b, 1) \wedge (1, 0) = (b, 0) \in I \times \{0\}$. Thus either $a \in I$ or $b \in I$.
(ii) We show that I is not a prime ideal of L_1. Suppose that I is a prime ideal of L_1. As $J = \{0\}$ and J is a prime ideal of L_2 and I is a prime ideal of L_1, we have $I \times \{0\}$ is a 2-absorbing ideal of L by Theorem 3.4, a contradiction to the hypothesis.
(2) $\Longrightarrow$ (1). Suppose that I is a weakly prime ideal of L_1, that is not a prime ideal and $J = \{0\}$ is a prime ideal of L_2. We show that $I \times \{0\}$ is a weakly 2-absorbing ideal of L. Suppose that for some $a, c, e \in L_1$ and $b, d, f \in L_2$, $(a, b) \wedge (c, d) \wedge (e, f) = (a \wedge c \wedge e, b \wedge d \wedge f) \in I \times \{0\} - \{(0, 0)\}$ then $a \wedge c \wedge e \in I$. Since I is a weakly prime ideal of L_1, we may assume $a \in I$. Since L_2 is an integral lattice, we may assume $d = 0$. Hence $(a, b) \wedge (c, d) = (a, b) \wedge (c, 0) = (a \wedge c, 0) \in I \times \{0\}$. Thus $I \times \{0\}$ is a weakly 2-absorbing ideal of L. We show that $I \times \{0\}$ is not a 2-absorbing ideal of L. Since I is a weakly prime ideal of L_1, that is not a prime ideal, there are $a, b \in L_1$ such that $a \wedge b = 0$ but neither $a \in I$ nor $b \in I$. Since $(a, 1) \wedge (b, 1) \wedge (1, 0) = (a \wedge b, 0) = (0, 0)$ and neither $(a, 1) \wedge (b, 1) = (a \wedge b, 1) \in I \times \{0\}$ nor $(a, 1) \wedge (1, 0) = (a, 0) \in I \times \{0\}$ nor $(b, 1) \wedge (1, 0) = (b, 0) \in I \times \{0\}$, we conclude that $I \times \{0\}$ is not a 2-absorbing ideal of L. ☐

The following Lemma is from Wasadikar and Gaikwad [5].

Lemma 3.1. *Let P_1 and P_2 be two distinct prime ideals of a lattice L, then $P_1 \cap P_2$ is a 2-absorbing ideal of L.*

In the following Theorem we prove that an ideal of a lattice which is a product of three lattices is either zero or 2-absorbing.

Theorem 3.6. *Let $L = L_1 \times L_2 \times L_3$ where L_1, L_2 and L_3 are bounded lattices. Then an ideal I of L is a weakly 2-absorbing ideal if and only if either $I = \{(0, 0, 0)\}$ or I is a 2-absorbing ideal of L.*

Proof. Let $I = I_1 \times I_2 \times I_3$ be a weakly 2-absorbing ideal of L. We may assume that $I \neq 0$. Suppose that $(0, 0, 0) \neq (p, q, r) \in I$. Then $(p, 1, 1) \wedge (1, q, 1) \wedge (1, 1, r) =$

$(p, q, r) \in I$. As I is a weakly 2-absorbing ideal, either $(p, 1, 1) \wedge (1, q, 1) = (p, q, 1) \in I$ or $(p, 1, 1) \wedge (1, 1, r) = (p, 1, r) \in I$ or $(1, q, 1) \wedge (1, 1, r) = (1, q, r) \in I$. Without loss of generality suppose that $(p, q, 1) \in I$. Then $I_3 = L_3$. Thus $I = I_1 \times I_2 \times L_3$.

We show that $I_1 \times I_2$ is a 2-absorbing ideal of $L_1 \times L_2$, hence I is a 2-absorbing ideal of L.

We first show that I_1 is a prime ideal of L_1 and I_2 is a prime ideal of L_2.

Let $a \wedge b \in I_1$ and $c \wedge d \in I_2$, for $a, b \in L_1$ and $c, d \in L_2$. Then $(a, 1, 1) \wedge (1, c \wedge d, 1) \wedge (b, 1, 1) = (a \wedge b, c \wedge d, 1) \in I = I_1 \times I_2 \times L_3$. As $I_2 \neq L_2$, $(a, 1, 1) \wedge (b, 1, 1) \notin I$. Hence either $(a, 1, 1) \wedge (1, c \wedge d, 1) = (a, c \wedge d, 1) \in I$ or $(1, c \wedge d, 1) \wedge (b, 1, 1) = (b, c \wedge d, 1) \in I$. That is either $a \in I_1$ or $b \in I_1$. Thus I_1 is a prime ideal of L_1. Similarly $(a \wedge b, 1, 1) \wedge (1, c, 1)(1, d, 1) = (a \wedge b, c \wedge d, 1) \in I = I_1 \times I_2 \times L_3$ and as $I_1 \neq L_1$, $(1, c, 1) \wedge (1, d, 1) \notin I$. We conclude that either $(a \wedge b, 1, 1) \wedge (1, c, 1) = (a \wedge b, c, 1) \in I$ or $(a \wedge b, 1, 1) \wedge (1, d, 1) = (a \wedge b, d, 1) \in I$. That is either $c \in I_2$ or $d \in I_2$. Thus I_2 is a prime ideal of L_2. Let $P = I_1 \times L_2$ and $Q = L_1 \times I_2$. As I_1 is a prime ideal of L_1 and I_2 is a prime ideal of L_2, P and Q are prime ideals of $L_1 \times L_2$. Thus $P \cap Q = I_1 \times I_2$ is a 2-absorbing ideal of $L_1 \times L_2$ by Lemma 3.1. Thus $I = I_1 \times I_2 \times L_3$ is a 2-absorbing ideal of $L = L_1 \times L_2 \times L_3$. Similarly, if $(p, 1, r) \in I$, we can show that $I = I_1 \times L_2 \times I_3$ is a 2-absorbing ideal of $L = L_1 \times L_2 \times L_3$ and if $(1, q, r) \in I$ we can show that $I = L_1 \times I_2 \times I_3$ is a 2-absorbing ideal of $L = L_1 \times L_2 \times L_3$.

Converse is trivial. □

By using the Theorem 3.6, we prove the following Theorem.

Theorem 3.7. *Let $L = L_1 \times L_2 \times L_3$ where L_1, L_2 and L_3 are bounded lattices. Let I_1 be a proper ideal of L_1, I_2 be an ideal of L_2 and I_3 be an ideal of L_3 such that $I = I_1 \times I_2 \times I_3 \neq \{(0, 0, 0)\}$. Then the following statements are equivalent:*

1. *$I = I_1 \times I_2 \times I_3$ is a weakly 2-absorbing ideal of L.*
2. *$I = I_1 \times I_2 \times I_3$ is a 2-absorbing ideal of L.*
3. *$I = I_1 \times L_2 \times L_3$ and I_1 is a 2-absorbing ideal of L_1 or $I = I_1 \times I_2 \times L_3$ such that I_1 is a prime ideal of L_1 and I_2 is a prime ideal of L_2 or $I = I_1 \times L_2 \times I_3$ such that I_1 is a prime ideal of L_1 and I_3 is a prime ideal of L_3.*

Proof. (1) $\Longrightarrow$ (2). Since I is a nonzero weakly 2-absorbing ideal, I is a 2-absorbing ideal of L by Theorem 3.6.

(2) $\Longrightarrow$ (3). Suppose that $I = I_1 \times I_2 \times I_3$ is a 2-absorbing ideal of L.

Since I_1 is a proper ideal of L_1 and if $I = I_1 \times I_2 \times I_3$ is a 2-absorbing ideal of L then I_1 is a 2-absorbing ideal of L_1. Since $I \neq \{(0, 0, 0)\}$, there is an element $(0, 0, 0) \neq (a, b, c) \in I$. Then $(a, 1, 1) \wedge (1, b, 1) \wedge (1, 1, c) = (a, b, c) \in I$, and hence $(a, 1, 1) \wedge (1, b, 1) = (a, b, 1) \in I$ or $(a, 1, 1) \wedge (1, 1, c) = (a, 1, c) \in I$ or $(1, b, 1) \wedge (1, 1, c) = (1, b, c) \in I$.

If $(a, b, 1) \in I = I_1 \times I_2 \times I_3$ then $1 \in I_3$ that is $I_3 = L_3$. If $(a, 1, c) \in I = I_1 \times I_2 \times I_3$ then $1 \in I_2$ that is $I_2 = L_2$. If $(1, b, c) \in I = I_1 \times I_2 \times I_3$ then $1 \in I_1$ that is $I_1 = L_1$.

As I_1 is a proper ideal of L_1, we have $I_1 \neq L_1$. Hence either $I_2 = L_2$ or $I_3 = L_3$.

(a). Assume that $I_2 \neq L_2$ and $I_3 = L_3$, hence $I = I_1 \times I_2 \times L_3$. We show that I_1 is a prime

ideal of L_1 and I_2 is a prime ideal of L_2. Let $a, b \in L_1$ be such that $a \wedge b \in I_1$ and let $c, d \in L_2$ be such that $c \wedge d \in I_2$. Then $(a, 1, 1) \wedge (1, c \wedge d, 1) \wedge (b, 1, 1) = (a \wedge b, c \wedge d, 1) \in I$. As $I_2 \neq L_2$, we have $(a, 1, 1) \wedge (b, 1, 1) \notin I$. Hence either $(a, 1, 1) \wedge (1, c \wedge d, 1) = (a, c \wedge d, 1) \in I$ or $(1, c \wedge d, 1) \wedge (b, 1, 1) = (b, c \wedge d, 1) \in I$. That is $a \in I_1$ or $b \in I_1$. Thus I_1 is a prime ideal of L_1. Similarly, I_2 is a prime ideal of L_2.
(b). Assume that $I_2 = L_2$ and $I_3 \neq L_3$, hence $I = I_1 \times L_2 \times I_3$. We show that I_1 is a prime ideal of L_1 and I_3 is a prime ideal of L_3. Let $a, b \in L_1$ such that $a \wedge b \in I_1$ and let $c, d \in L_3$ such that $c \wedge d \in I_3$. Then $(a, 1, 1) \wedge (1, 1, c \wedge d) \wedge (b, 1, 1) = (a \wedge b, 1, c \wedge d) \in I$. As $I_3 \neq L_3$, $(a, 1, 1) \wedge (b, 1, 1) = (a \wedge b, 1, 1) \notin I$. We conclude that either $(a, 1, 1) \wedge (1, 1, c \wedge d) = (a, 1, c \wedge d) \in I$ or $(1, 1, c \wedge d) \wedge (b, 1, 1) = (b, 1, c \wedge d) \in I$. That is either $a \in I_1$ or $b \in I_1$. Thus I_1 is a prime ideal of L_1. Similarly, I_3 is a prime ideal of L_3.
(3) $\implies$ (1). Suppose that $I = I_1 \times L_2 \times L_3$ and I_1 is a 2-absorbing ideal of L_1 or $I = I_1 \times I_2 \times L_3$ such that I_1 is a prime ideal of L_1 and I_2 is a prime ideal of L_2 or $I = I_1 \times L_2 \times I_3$ such that I_1 is a prime ideal of L_1 and I_3 is a prime ideal of L_3.
Case 1: Suppose that $I = I_1 \times L_2 \times L_3$ and I_1 is a 2-absorbing ideal of L_1. We show that I is a 2-absorbing ideal of L. Suppose that for $a_i \in L_1$, $b_i \in L_2$ and $c_i \in L_3$ where $i = 1, 2, 3$ with $(a_1, b_1, c_1) \wedge (a_2, b_2, c_2) \wedge (a_3, b_3, c_3) \in I = I_1 \times L_2 \times L_3$. Then $(a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3, c_1 \wedge c_2 \wedge c_3) \in I = I_2 \times L_2 \times L_3$. Since I_1 is a 2-absorbing ideal of L_1, either $a_1 \wedge a_2 \in I_1$ or $a_1 \wedge a_3 \in I_1$ or $a_2 \wedge a_3 \in I_1$. Without loss of generality, suppose that $a_1 \wedge a_2 \in I_1$. Then $(a_1 \wedge a_2, b_1 \wedge b_2, c_1 \wedge c_2) \in I = I_1 \times L_2 \times L_3$. That is $(a_1, b_1, c_1) \wedge (a_2, b_2, c_2) \in I = I_1 \times L_2 \times L_3$. Thus $I = I_1 \times L_2 \times L_3$ is a 2-absorbing ideal of L.
Case 2: Suppose that $I = I_1 \times I_2 \times L_3$ such that I_1 is a prime ideal of L_1 and I_2 is a prime ideal of L_2. We show that I is a 2-absorbing ideal of L. Suppose that for $a_i \in L_1$, $b_i \in L_2$, $c_i \in L_3$ where $i = 1, 2, 3$ with $(a_1, b_1, c_1) \wedge (a_2, b_2, c_2) \wedge (a_3, b_3, c_3) \in I = I_1 \times I_2 \times L_3$. Then $(a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3, c_1 \wedge c_2 \wedge c_3)$ Since I_1 is a prime ideal of L_1, one of the $a_i's$ is in I_1, say a_1 and since I_2 is a prime ideal of L_2, one of the $b_i's$ is in I_2, say b_2. Therefore $a_1 \wedge a_2 \in I_1$ and $b_1 \wedge b_2 \in I_2$, implies $(a_1 \wedge a_2, b_1 \wedge b_2, c_1 \wedge c_2) \in I_1 \times I_2 \times L_3$. Hence $(a_1, b_1, c_1) \wedge (a_2, b_2, c_2) \in I = I_1 \times I_2 \times L_3$.

Thus $I = I_1 \times I_2 \times L_3$ is a 2-absorbing ideal of L.
Case 3: Suppose that $I = I_1 \times L_2 \times I_3$ such that I_1 is a prime ideal of L_1 and I_3 is a prime ideal of L_3. Similarly as in case 1 we can show that I is a 2-absorbing ideal of L.

That is if L is one of the given three forms, then we have I is a 2-absorbing ideal of L.

As every 2-absorbing ideal is a weakly 2-absorbing ideal, I is a weakly 2-absorbing ideal of L. □

Now we give a characterization of a 2-absorbing ideal in the product of lattices.

Theorem 3.8. *Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices with one. Let I be a proper ideal of L. Then the following statements are equivalent:*
1. *I is a 2-absorbing ideal of L.*

2. *Either $I = I_1 \times L_2$ for some 2-absorbing ideal I_1 of L_1 or $I = L_1 \times I_2$ for some 2-absorbing ideal I_2 of L_2 or $I = I_1 \times I_2$ for some prime ideal I_1 of L_1 and some prime ideal I_2 of L_2.*

Proof. (1) $\Longrightarrow$ (2). Suppose that I is a 2-absorbing ideal of L. Then $I = I_1 \times I_2$ for some ideal I_1 of L_1 and some ideal I_2 of L_2. If $I_2 = L_2$ then $I_1 \neq L_1$. Thus $I = I_1 \times L_2$. As $I = I_1 \times L_2$ is a 2-absorbing ideal of L, I_1 is a 2-absorbing ideal of L_1, by Theorem 3.3. If $I_1 = L_1$ then $I_2 \neq L_2$. Thus $I = L_1 \times I_2$. As $I = L_1 \times I_2$ is a 2-absorbing ideal of L, I_2 is a 2-absorbing ideal of L_2, by Theorem 3.3. Now suppose that $I = I_1 \times I_2$ and $I_1 \neq L_1, I_2 \neq L_2$. Suppose that I_1 is not a prime ideal of L_1, then there are $a, b \in L_1$ such that $a \wedge b \in I_1$ but neither $a \in I_1$ nor $b \in I_1$. Let $x = (a, 1)$, $y = (1, 0)$ and $c = (b, 1)$. Then $x \wedge y \wedge c = (a \wedge b, 0) \in I$ but neither $x \wedge y = (a, 0) \in I$ nor $x \wedge c = (a \wedge b, 1) \in I$ nor $y \wedge c = (b, 0) \in I$, which is a contradiction. Thus I_1 is a prime ideal of L_1. Similarly, I_2 is a prime ideal of L_2.
(2) $\Longrightarrow$ (1). Case 1: Suppose that $I = I_1 \times L_2$ for some 2-absorbing prime ideal I_1 of L_1. Let $(a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I_1 \times L_2$. Then $a_1 \wedge a_2 \wedge a_3 \in I_1$. As I_1 is a 2-absorbing ideal of L_1, we have either $a_1 \wedge a_2 \in I_1$ or $a_1 \wedge a_3 \in I_1$ or $a_2 \wedge a_3 \in I_1$. That is either $(a_1, b_1) \wedge (a_2, b_2) \in I_1 \times L_2$ or $(a_1, b_1) \wedge (a_3, b_3) \in I_1 \times L_2$ or $(a_2, b_2) \wedge (a_3, b_3) \in I_1 \times L_2$. Thus $I = I_1 \times L_2$ is a 2-absorbing ideal of L.
Case 2: Similarly, $L_1 \times I_2$ is a 2-absorbing ideal of L.
Case 3: Suppose that $I = I_1 \times I_2$ for some prime ideal I_1 of L_1 and some prime ideal I_2 of L_2. Then $P = I_1 \times L_2$ and $Q = L_1 \times I_2$ are prime ideals of L. Hence $P \cap Q = I_1 \times I_2$. Thus $I = I_1 \times I_2$ is a 2-absorbing ideal of L, by Lemma 3.1. □

The following Theorem is a generalization of the Theorem 3.8.

Theorem 3.9. *Let $L = L_1 \times L_2 \cdots \times L_n$, where $2 \leq n < \infty$, and $L_1, L_2, \cdots L_n$ are lattices. Let I be a proper ideal of L. Then the following statements are equivalent.*
1. *I is a 2-absorbing ideal of L.*
2. *Either $I = \prod_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, \cdots n\}$, I_k is a 2-absorbing ideal of L_k, and $I_t = L_t$ for every $t \in \{1, 2, \cdots n\} \backslash \{k\}$ or $I = \prod_{t=1}^{n} I_t$ such that for some $k, m \in \{1, 2, \cdots n\}$, I_k is a prime ideal of L_k, I_m is a prime ideal of L_m, and $I_t \neq L_t$ for every $t \in \{1, 2, \cdots n\} \backslash \{k, m\}$.*

Proof. (1) $\Longleftrightarrow$ (2) We prove this Theorem by induction method on n. Assume $n = 2$. Then by Theorem 3.8, the result holds. Thus suppose that $3 \leq n < \infty$ and assume that the result is valid when $K = L_1 \times L_2 \cdots L_{n-1}$. Now we prove the result when $L = K \times L_n$. By Theorem 3.8, I is a 2-absorbing ideal of L if and only if either $I = A \times L_n$ for some 2-absorbing ideal A of K or $I = K \times A_n$ for some 2-absorbing ideal A_n of L_n or $I = A \times A_n$ for some prime ideal A of K and some prime ideal A_n of L_n. Now observe that a proper ideal B of K is a prime ideal of K if and only if $B = \prod_{t=1}^{n-1} I_t$ such that for some $k \in \{1, 2, \cdots n-1\}$, I_k is a prime ideal of L_k, and $I_t \neq L_t$ for every $t \in \{1, 2, \cdots n-1\} \backslash \{k, m\}$. □

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Separability in algebra and category theory

Abstract: Separable field extensions are essentially known since the 19th century and their formal definition was given by Ernst Steinitz in 1910. In this survey we first recall this notion and equivalent characterisations. Then we outline how these were extended to more general structures, leading to separable algebras (over rings), Frobenius algebras, (non associative) Azumaya algebras, coalgebras, Hopf algebras, and eventually to separable functors. The purpose of the talk is to demonstrate that the development of new notions and definitions can lead to simpler formulations and to a deeper understanding of the original concepts. The formalism also applies to algebras and coalgebras over semirings and S-acts (transition systems).

Preliminaries

The investigation of the interplay between (roots of) polynomials over the rationals and (automorphisms of) extensions of the rationals was started by Joseph-Louis Lagrange (1736 -1813). Through the work of Évariste Galois (1811–1832) it became evident that this was the key to an interesting deep relationship between finite extensions of the rationals and the theory of finite groups. The first presentation of Galois' ideas in a textbook was given 1866 by Joseph Alfred Serret (1819–1885). At that time, the abstract notion of a *field* was not yet available in algebra. The (German) name *Körper* (Engl. *body*) was coined by Richard Dedekind in 1871, meaning substructures of the complex numbers. His intention of choosing this name was to signify – as in the natural sciences, geometry, and human society – a system with certain completeness, seclusion, and perfection. He provided the fundamentals for *linear algebra* over a *Körper* like linear dependence, basis, dimensions as well as trace and norm of finite extensions.

In 1893, Heinrich M. Weber [78] defined the abstract notion of a *Körper* (as a set allowing for addition and multiplication subject to certain conditions) to give the right frame to Galois theory. Eliakim H. Moore used in [59] the English word *field* as synonym for *endliche Körper* in the sense of Weber. Nowadays *field* and *Körper* are synonyms without finiteness restrictions.

The abstract *theory of fields* was initiated by Ernst Steinitz in 1910 [72]. Systematically developing the axioms of (commutative) fields he introduced the notions of *prime field*, *algebraic closure*, and *field extensions of first (resp. second) kind* which later on were called *separable* (resp. *inseparable*) extensions in van der Waerden's textbook from 1930.

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New perspectives in the study of field extensions were obtained by applying *tensor products*. These arose (under different labels) in the late 19th century in physics and mathematics for vector spaces over the real or complex numbers (e.g. Gibbs [5]). The crucial step in extending tensor products to *abelian groups* was made by Hassler Whitney in [79], 1938. From this, tensor products of modules over rings (and fields) are obtained by suitable coequalisers.

In this survey, we shall first recall the properties of separable field extensions, and then generalisations deduced from them. Recall that a field extension $L : K$ is called *separable* provided every element of L is a root of an irreducible polynomial with coefficients in K which does not have multiple roots.

For any normal extension $L : K$, the automorphism group $G := Aut(L : K)$ acts on L thus making L a module over the group algebra $K[G]$ and a comodule over the dual coalgebra $K[G]^*$. Then $L : K$ is separable provided $L \simeq K[G]$ as $K[G]$-module.

For any field extension $L : K$, L is a vector space over K and any L-vector space Y is given by a K-linear map $\varrho\colon L \otimes_K Y \to Y$. Furthermore, for any K-vector space X, $L \otimes_K X$ is an L-vector space by the action

$$m \otimes X\colon L \otimes_K L \otimes_K X \to L \otimes_K X\,, \quad a \otimes b \otimes x \mapsto ab \otimes x\,,$$

where $m\colon L \otimes_K L \to L$ denotes the multiplication in L. Denoting by $\mathbb{V}_k$ the category of vector spaces over any field k, this gives rise to the functors

$$\begin{aligned} L \otimes_K -\colon &\quad \mathbb{V}_K \to \mathbb{V}_K, \quad X \mapsto L \otimes_K X\,,\\ \phi_L\colon &\quad \mathbb{V}_K \to \mathbb{V}_L, \quad X \mapsto (L \otimes_K X, m \otimes X)\,,\\ U_L\colon &\quad \mathbb{V}_L \to \mathbb{V}_K, \quad (Y, L \otimes_K Y \to Y) \mapsto Y\,,\end{aligned}$$

and ϕ_L (the *free functor*) is left adjoint to U_L (the *forgetful functor*). Now, separability of $L : K$ is equivalent to require that, for any $V, N \in \mathbb{V}_L$, the canonical map

$$Hom_L(V, N) \to Hom_K(U_L(V), U_L(N))$$

is a (naturally) splitting monomorphism.

The setting just described can readily be transferred to general categories, replacing the functor $L \otimes_K -$ by a monad on any category $\mathbb{A}$ and defining separable and Frobenius functors (and (co)monads). This is done in Section 2 and, for the convenience of the reader, basic notions from category theory are recalled in Appendix 7. In the ensuing sections, the categorical notions are applied to generalise field extensions by replacing

(i) the field L by a (non-) associative algebra A (Section 3);
(ii) the base field K by a (non-) commutative ring R (Subsection 3.7);
(iii) the field L by a coalgebra C (Section 4);
(iv) the base field K by a semiring R (Section 5);
(v) the base field K by a set A (Section 6);

(vi) the bialgebra $K[G]$ by any Hopf algebra (Subsection 4.3).

In the theory of separability for algebras, *derivations* also play a mayor role (e.g. in [22, 46]). However, this aspect is not addressed in the present survey.

1 Separable and normal fields

A *field* is a triple $K = (K, +, \cdot)$ where K is a set, $(K, +)$ is an abelian group with neutral element 0,

$$\cdot : K \times K \to K \tag{1}$$

is a bilinear map making $(K \setminus \{0\}, \cdot)$ an abelian group with unit 1, and the relation between $+$ and $\cdot$ is given by distributivity.

$K[X]$ will denote the ring of polynomials with coefficients in K.

A *morphism* between two fields K, K' is a map $K \to K'$ respecting addition, multiplication, the neutral element, and the unit.

As prototypes of fields we have the rationals $\mathbb{Q}$ and the factor rings $\mathbb{Z}/p\mathbb{Z}$ of the integers $\mathbb{Z}$, for any prime number p. There are no field morphisms between these two types.

1.1 Field extensions

A subset K of a field L is called a subfield, if K contains 1 and is closed under subtraction and division in L. This setting is also called a *field extension* and denoted by $L : K$. By an *intermediate field* M of $L : K$ we mean a subfield of L containing K as a subfield. For a subset $S \subset L$, the smallest intermediate field of $L : K$ containing S is denoted by $K(S)$, and for $a \in L$ we write $K(\{a\}) = K(a)$.

$L : K$ is called a *finite extension*, if $\dim_K L \leq \infty$ and this dimension is the *degree* $[L : K]$ of the extension.

An element $a \in L$ is called *algebraic over* K, if there is a non-vanishing polynomial $f \in K[X]$ with $f(a) = 0$. The monic polynomial of smallest degree with this property is called the *minimal polynomial* of a over K, denoted by $Min(a : K)$; its degree is $[K(a) : K]$.

The extension $L : K$ is called *algebraic*, if all $a \in L$ are algebraic over K. The field K is called *algebraically closed*, if K has no proper algebraic extension.

1.2 K-morphisms

If $L : K$ and $Q : K$ are field extensions, and $\varphi : L \to Q$ is a field homomorphism such that $\varphi(k) = k$ for all $k \in K$, then φ is called a *K-morphism* (also *K-isomorphism*, e.g. in [13]).

A theorem of Dedekind (from 1871, [47]) says that a family of pairwise distinct K-morphisms $L \to Q$ are linearly independent over Q (as elements of $Hom_K(L, Q)$).

Evidently, the bijective K-morphisms $L \to L$ form a group, the *automorphism group* of $L : K$,

$$Aut(L : K) = \{\varphi : L \to L \mid \varphi \text{ is a } K\text{-isomorphism}\}\,,$$

which acts on L. If $L : K$ is algebraic, its orbits on L are finite (consisting of roots of some irreducible polynomial).

For any subset (subgroup) $H \subseteq Aut(L : K)$, the invariant elements

$$Fix(L : H) := \{a \in L \mid h(a) = a \text{ for any } h \in H\}$$

form an intermediate field of $L : K$.

The main concern of studying field extensions is to find roots for polynomials and the next two results give a basic answer to this problem.

1.3 Splitting fields

Let K be a field.

(i) (Kronecker 1887): For any polynomial $f \in K[X]$, there is a smallest extension $L : K$ such that f splits completely in L into a product of linear polynomials and a constant.

L is generated as a field by the roots of f and is unique up to K-isomorphisms. It is called the splitting field of f over K.

(ii) (Steinitz 1910): For any set of polynomials $S \subseteq K[X]$, there is a smallest extension $L : K$ such that any $f \in S$ splits completely in L into a product of linear polynomials and a constant.

For $S = K[X]$, this gives an algebraic extension $\widehat{K}$ of K which is algebraically closed.

As easily seen, the number of roots of an irreducible $f \in K[X]$ may be smaller then the dimension of the splitting field $L : K$. These numbers are equal for the

1.4 Separable polynomials

An irreducible polynomial $f \in K[X]$ is called *separable*, if it has no double roots in an algebraic closure $\widehat{K}$ of K. An algebraic field extension $L : K$ is said to be *separable* provided, for every $a \in L$, the minimal polynomial $Min(a : K)$ is separable.

For an irreducible $f \in K[X]$, the following are equivalent:

(a) *f is separable;*

(b) *for any field extension $L : K$ and $a \in L$, $(X - a)^2$ does not divide f in $L[X]$;*

(c) *there is an extension $L : K$ such that f has* $\deg(f)$ *roots in L;*

(d) *the number of distinct roots of f is equal to the degree of f.*

1.5 Normal extensions

An algebraic field extension $L : K$ is called *normal* if, for every $a \in L$, the minimal polynomial $Min(a : K)$ splits completely into linear factors over L. In this case, $L : M$ is normal for each intermediate field M of $L : K$.

Let K be a field with algebraic closure $\widehat{K}$ and L an intermediate field of $\widehat{K} : K$. Then the following are equivalent:

(a) *$L : K$ is normal;*

(b) *L is the splitting field of a set of polynomials in $K[X]$;*

(c) *for any $\varphi \in Aut(\widehat{K} : K)$, $\varphi(L) \subseteq L$.*

Combining the preceding notions, we get extensions of particular interest:

1.6 Galois extensions

A field extension $L : K$ is said to be *Galois* provided it is separable and normal; $Aut(L : K)$ is called its *Galois group*. *For a finite field extension $L : K$, the following are equivalent:*

(a) *$L : K$ is Galois;*

(b) *L is the splitting field of a separable polynomial in $K[X]$;*

(c) $[L : K] = |Aut(L : K)|$*;*

(d) $Fix(L : Aut(L : K)) = K$.

In view of the observations made above it is not difficult now to prove our next result. It shows the close connection between the structures of groups and field extensions. For example, it implies that polynomials of degree 5 need not be *solvable by radicals* since the corresponding Galois groups need not be *solvable*, that is, need not have a subnormal series with abelian factor groups.

1.7 Galois correspondence

For a finite Galois extension $L : K$, there is an (order reversing) bijection between the sets (lattices)

$$\mathbb{F} := \{\text{intermediate fields of } L : K\}\ ,$$

$$\mathbb{G} := \{\text{subgroups of } Aut(L : K)\}\ ,$$

given by

$$Aut(L:-):\mathbb{F}\to\mathbb{G},\quad M\mapsto Aut(L:M)\,,$$
$$Fix(L:-):\mathbb{G}\to\mathbb{F},\quad H\mapsto Fix(L,H)\,.$$

Notice that a Galois extension $L : K$ need not be finite dimensional. In case it is not, there is a bijective correspondence between $\mathbb{F}$, the intermediate fields of $L : K$, and those subgroups of $Aut(L : K)$, which are closed in the topology on $Aut(L : K)$ induced by the pointwise convergence on L. Then $Aut(L : K)$ is a projective limit of the finite groups $Aut(M : K)$, where the intermediate fields M are finite Galois extensions of K (see [44]).

1.8 Trace form

For any finite field extension $L : K$, the K-endomorphism ring of the vector space L, $\mathrm{End}_K(L)$, is isomorphic to the $n \times n$-matrix ring over K, $n = [L : K]$. The trace function on this ring (sum of diagonal elements) is K-linear and thus provides a K-linear map $Tr\colon \mathrm{End}_K(L) \to K$ satisfying

$$Tr(f\circ g) = Tr(g\circ f) \text{ for any } f,g\in \mathrm{End}_K(L)\,.$$

The (left) multiplication by any $a \in L$,

$$\lambda_a\colon L\to L\,,\quad x\mapsto ax\,,$$

is in $\mathrm{End}_K(L)$ and for $a, b \in L$, $\lambda_{ab} = \lambda_a \circ \lambda_b$, thus yielding an (injective) K-algebra morphism

$$\lambda\colon L\to \mathrm{End}_K(L)\,,\quad a\mapsto \lambda_a\,.$$

Composing these maps we get the K-linear map

$$\mathrm{tr}\colon L\xrightarrow{\lambda}\mathrm{End}_K(L)\xrightarrow{Tr} K\,, \tag{2}$$

with the properties

$$\mathrm{tr}(ab) = Tr(\lambda_a\circ\lambda_b) = Tr(\lambda_b\circ\lambda_a) = \mathrm{tr}(ba)\,. \tag{3}$$

For separable extension $L : K$, with algebraic closure $\widehat{K} : K$, and $a \in L$,

$$\mathrm{tr}(a) = \sum\{\sigma(a)\mid \sigma\colon L\to\widehat{K} \text{ is a } K\text{-morphism}\}\,, \tag{4}$$

and it follows from Dedekind's theorem (see 1.2) that $\mathrm{tr}(a) \neq 0$.

If $L : K$ is Galois, $a \in L$, and $G = Aut(L : K)$,

$$\mathrm{tr}(a) = \sum_{\sigma\in G}\sigma(a)\,. \tag{5}$$

1.9 The dual space

For any field extension $L : K$, $L^* = Hom_K(L, K)$ is an L-vector space with the action of $b \in L$ on $f \in L^*$ given by $b \cdot f(-) := f(b-)$.

Fixing $a \in L$, $\mathrm{tr}(a-)\colon L \to K$, $x \mapsto \mathrm{tr}(ax)$, is in L^* and the map

$$\psi\colon L \to L^* , \quad a \mapsto \mathrm{tr}(a-) ,$$

is L-linear since $b \cdot \mathrm{tr}(ax) = \mathrm{tr}(b(ax)) = \mathrm{tr}((ab)x)$, that is,

$$b \cdot \mathrm{tr}(a-) = \mathrm{tr}(ba-) .$$

If $L : K$ is finite and separable, $\mathrm{tr}(a)$ is nonzero for $a \in L$ (see (4)) and hence ψ is injective, and in fact bijective, since $\dim_K L = \dim_K L^*$.

These are the ingredients for the

Proposition 1.1. *For a finite field extension $L : K$, there are equivalent:*

(a) *$L : K$ is separable;*

(b) *$tr\colon L \to K$ is not zero;*

(c) *$\psi\colon L \to L^*$, $a \mapsto tr(a-)$, is an isomorphism (and is L-linear).*

Notice that so far our results are essentially obtained with the techniques available in the 19th century. The way of looking at the material was greatly expanded when the following notion came into play.

1.10 Tensor product of modules

Let R be a commutative ring. The *tensor product* of R-modules N and M is a pair $(N \otimes_R M, \pi)$ where $N \otimes_R M$ is an R-module and $\pi\colon N \times M \to L \otimes_R M$ is an R-bilinear map such that, for any R-module G and R-bilinear map $\beta\colon N \times M \to G$, there exists a unique R-module homomorphism $\widetilde{\beta}\colon N \otimes_R M \to G$ with commutative diagram

$$\begin{tikzcd} N \times M \arrow[r, "\pi"] \arrow[dr, "\beta"'] & N \otimes_R M \arrow[d, "\widetilde{\beta}"] \\ & G. \end{tikzcd}$$

The tensor product of R-linear maps $f\colon N \to N'$ and $g\colon M \to M'$ between R-modules, is defined as $f \otimes g\colon N \otimes_R M \to N' \otimes_R M'$ by putting

$$f \otimes g(n \otimes m) := f(n) \otimes g(m) , \quad \text{for } n \in N,\ m \in M ,$$

and showing that this defines in fact a map on $N \otimes_R M$. For the identity $1_N\colon N \to N$, it is customary just to write $1_N \otimes g = N \otimes g$. We often delete the suffix of $\otimes_R$ (in particular in formulas) and just write $\otimes$ if no ambiguity arises.

1.11 Tensor product and field extensions

For a field extension $L : K$, the product $m\colon L \otimes_K L \to L$ and unit $\iota\colon K \to L$ are K-linear maps with commutative diagrams

$$\begin{tikzcd} L\otimes L\otimes L \arrow[r, "m\otimes L"] \arrow[d, "L\otimes m"'] & L\otimes L \arrow[d, "m"] \\ L\otimes L \arrow[r, "m"] & L, \end{tikzcd} \qquad \begin{tikzcd} K\otimes L \arrow[r, "\iota\otimes L"] \arrow[dr, "="'] & L\otimes L \arrow[d, "m"] & L\otimes K \arrow[l, "L\otimes\iota"'] \arrow[dl, "="] \\ & L & \end{tikzcd} . \tag{6}$$

For two field extensions $L : K$ and $Q : K$, a multiplication on $Q \otimes_K L$ is defined by putting, for $q_1, q_2 \in Q$ and $l_1, l_2 \in L$,

$$(q_1 \otimes l_1) \cdot (q_2 \otimes l_2) = q_1q_2 \otimes l_1l_2 \, .$$

This makes $Q \otimes_K L$ a K-algebra which need not be a field, in particular, it may allow for nilpotent elements.

The set of all nilpotent elements of a commutative finite dimensional K-algebra A is called the *nil radical* and is denoted by $Nil(A)$. The structure theorem says that $Nil(A) = 0$ if and only if A is a (finite) direct product of fields. The relevance of nilpotency for separability becomes evident in the following result (e.g. [87, Section 28]).

Proposition 1.2. *Let $L : K$ be an algebraic field extension and assume $a \in L$ is not separable over K. Then there exists some field extension $Q : K$ such that $L \otimes_K Q$ has non-zero nilpotent elements.*

Proof. Since over fields with zero characteristic all extensions are separable, we have to consider the case $char(K) = p \neq 0$. Let $f \in K[X]$ be the minimal polynomial of a over K and Q its splitting field. Then there exists a polynomial

$$h(X) = \textstyle\sum_{i=1}^r b_iX^i \quad \text{with} \quad b_i \in Q,\, b_r \neq 0, \quad \text{and} \quad f(X) = h(X)^p.$$

Since $r < n$, the elements $1, a, \cdots, a^r$ are linearly independent over K, hence

$$0 \neq c = \sum_{j=1}^{r} a^j \otimes b_j \in L \otimes_K Q \, , \quad \text{and}$$

$$c^p = (\sum_{j=1}^{r} a^j \otimes b_j)^p = \sum_{j=1}^{r} a^{jp} \otimes b_j^p = \sum_{j=1}^{r} (b_ja^j)^p \otimes 1 = h(a)^p \otimes 1 = 0 \, .$$

So $c \in L \otimes_K Q$ is non-zero and nilpotent □

Remark. A more general version of Proposition 1.2 can be given in terms of radicals, e.g. [69, Chapitre 2]: for any K-algebra A, define the *prime radical* $rad(A)$ as intersection of all prime ideals of A. Then, for any field extension $L : K$,

$$rad(A \otimes_K L) \cap A = rad(A) \, ,$$

and if $L : K$ is separable,

$$rad(A \otimes_K L) = rad(A) \otimes_K L\,.$$

Notice that for commutative algebras A, $rad(A) = Nil(A)$.

From these observations we get:

Proposition 1.3. *For a finite field extension $L : K$, the following properties are equivalent:*

(a) *$L : K$ is separable;*
(b) *for any field extension $Q : K$, $Nil(L \otimes_K Q) = 0$;*
(c) *$Nil(L \otimes_K L) = 0$;*
(d) *L is projective as an $L \otimes_K L$-module;*
(e) *$m\colon L \otimes_K L \to L$ is split by some (L, L)-bimodule morphism $\delta\colon L \to L \otimes_K L$;*
(f) *there exist $e \in L \otimes_K L$ with $ae = ea$, for any $a \in L$, and $m(e) = 1$ (choose $e = \delta(1)$, separability idempotent).*

By the structure theorem mentioned above, $Nil(L \otimes_K L) = 0$ implies that $L \otimes_K L$ is a semisimple algebra and hence all $L \otimes_K L$-modules are projective. The last two characterisations are just module theoretic variations of projectivity in the category of $L \otimes L$-modules.

The map δ in (e) opens the view to a new structure:

1.12 Coproduct on L

If $L : K$ is separable, then $\delta\colon L \to L \otimes_K L$ in (e) of the above proposition is left and right L-linear and this is expressed by commutativity of the diagrams

$$\begin{tikzcd} L \otimes L \arrow[r, "m"] \arrow[d, "\delta\otimes L"'] & L \arrow[d, "\delta"] \\ L \otimes L \otimes L \arrow[r, "L\otimes m"] & L \otimes L\,, \end{tikzcd} \qquad \begin{tikzcd} L \otimes L \arrow[r, "L\otimes\delta"] \arrow[d, "m"'] & L \otimes L \otimes L \arrow[d, "m\otimes L"] \\ L \arrow[r, "\delta"] & L \otimes L\,. \end{tikzcd} \tag{7}$$

These relations between m and δ are called *Frobenius conditions*.

Moreover, $m \circ \delta = 1_L$, and an easy argument shows that these imply commutativity of the diagram (*coassociativity*)

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L \otimes L \arrow[d, "L\otimes\delta"] \\ L \otimes L \arrow[r, "\delta\otimes L"] & L \otimes L \otimes L\,. \end{tikzcd} \tag{8}$$

There is yet another way to define a coproduct on L.

1.13 Dual of an algebra (coalgebra)

For any finite field extension $L : K$, applying $(-)^* = Hom_K(-, K)$ to $m\colon L \otimes_K L \to L$ and $\iota\colon K \to L$, yields K-linear maps

$$L^* \xrightarrow{m^*} (L \otimes_K L)^* \simeq L^* \otimes_K L^* , \qquad L^* \xrightarrow{\iota^*} K .$$

Given a K-isomorphism $\psi\colon L \to L^*$ leads to K-linear maps

$$\delta\colon L \to L \otimes_K L , \qquad \varepsilon\colon L \to K , \tag{9}$$

by the diagrams

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\psi"'] & L \otimes_K L \\ L^* \arrow[r, "m^*"] & L^* \otimes_K L^* , \arrow[u, "\psi^{-1}\otimes\psi^{-1}"'] \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, "\psi"'] \arrow[dr, "\varepsilon"] & \\ L^* \arrow[r, "\iota^*"] & K, \end{tikzcd} \tag{10}$$

and exploiting associativity of m and unitality yields commutativity of the diagrams (dual to (6))

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L \otimes L \arrow[d, "L\otimes\delta"] \\ L \otimes L \arrow[r, "\delta\otimes L"] & L \otimes L \otimes L , \end{tikzcd} \qquad \begin{tikzcd} L \arrow[dr, "="'] & L \otimes L \arrow[l, "L\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes L"] & L \arrow[dl, "="] \\ & L \arrow[u, "\delta"] & \end{tikzcd} , \tag{11}$$

which describe *coassociativity* and *counitality*, respectively.

In case $\psi\colon L \to L^*$ is even L-linear (e.g. Proposition 1.1), the Frobenius conditions (7) are satisfied for (L, m, δ).

1.14 Tensor functor

For a field extension $L : K$, the L-vector spaces can be defined as K-vector spaces V with K-linear maps $\varrho\colon L \otimes_K V \to V$ subject to associativity and unitality conditions, that is, commutativity of the diagrams

$$\begin{tikzcd} L \otimes L \otimes V \arrow[r, "m\otimes V"] \arrow[d, "L\otimes\varrho"'] & L \otimes V \arrow[d, "\varrho"] \\ L \otimes V \arrow[r, "\varrho"] & V, \end{tikzcd} \qquad \begin{tikzcd} L \otimes V \arrow[r, "\varrho"] & V \\ K \otimes V \arrow[u, "\iota\otimes V"] \arrow[ur, "="'] & \end{tikzcd} .$$

In particular, $L \otimes_K V$ is always an L-vector space by

$$m \otimes V\colon L \otimes L \otimes V \to L \otimes V ,$$

and this leads to the *extension of scalars* functor

$$\Phi_L\colon \mathbb{V}_K \to \mathbb{V}_L, \qquad V \mapsto (L \otimes_K V, m \otimes V)\,,$$
$$V \xrightarrow{f} V' \mapsto L \otimes_K V \xrightarrow{L\otimes f} L \otimes_K V'\,;$$

there is a *restriction of scalars* (or *forgetful*) functor

$$U_L\colon \mathbb{V}_L \to \mathbb{V}_K\,, \quad (V, \varrho) \mapsto V\,,$$

leaving morphisms unchanged. These form an *adjoint pair* of functors by the bijection, for $V \in \mathbb{V}_K$, $N \in \mathbb{V}_L$,

$$Hom_L(L \otimes_K V, N) \simeq Hom_K(V, U_L(N))\,, \quad f \mapsto f \circ (\iota \otimes V)\,.$$

It is customary to write $U_L(N) = N$ if no confusion occurs.

Splitting of U_L. Let $L : K$ be a finite separable field extension. For (V, ϱ), (N, ϱ') in $\mathbb{V}_L$, the forgetful functor $U_L\colon \mathbb{V}_L \to \mathbb{V}_K$ provides the canonical map (writing $U_L(V) = V$ etc.)

$$\Phi^U_{V,N}\colon Hom_L(V, N) \to Hom_K(U_L(V), U_L(N)) = Hom_K(V, N)\,. \tag{12}$$

Since (L, m, δ) satisfies the Frobenius conditions, it is not difficult to show that for the L-linear map

$$\omega\colon V \xrightarrow{\iota\otimes V} L \otimes V \xrightarrow{\delta\otimes V} L \otimes L \otimes V \xrightarrow{L\otimes\varrho} L \otimes V\,, \tag{13}$$

one has $\varrho \circ \omega = 1_V$. Now define the map (natural in V and N)

$$\Psi^U_{V,N}\colon Hom_K(V, N) \to Hom_L(V, N)\,,$$
$$V \xrightarrow{f} N \quad \mapsto \quad V \xrightarrow{\omega} L \otimes_K V \xrightarrow{L\otimes f} L \otimes_K N \xrightarrow{\varrho'} N\,.$$

If f happens to be L-linear one easily sees that $\Psi^U_{V,N}(f) = f$. This proves one implication of the

Proposition 1.4. *For a field extension $L : K$ there are equivalent:*

(a) *$L : K$ is separable;*

(b) *$\Phi^U_{V,N}$ is a naturally split monomorphism.*

1.15 Group algebra and its dual

Let G be any group with unit e and K a field (or commutative ring). The group algebra $K[G]$ is defined as a K-vector space with basis set G and product (multiplication) with unit

$$m_G\colon K[G] \otimes_K K[G] \to K[G],\ g \otimes h \mapsto gh\,, \quad \iota\colon K \to K[G],\ k \mapsto k{\cdot}e\,,$$

$K[G]$ also allows for a coproduct with counit

$$\delta_G\colon K[G] \to K[G] \otimes_K K[G],\ g \mapsto g \otimes g\,, \quad \varepsilon\colon K[G] \to K,\ g \mapsto \delta_{g,e}\,,$$

where $\delta_{g,e}$ denotes the Kronecker symbol. One has

$$(1_{K[G]} \otimes \varepsilon)\delta_G = 1_{K[G]} = (\varepsilon \otimes 1_{K[G]})\delta_G\,,$$

and for $u, v \in K[G]$, $\delta_G(uv) = \delta_G(u) \cdot \delta_G(v)$.

This shows that $(K[G], m_G, \delta_G)$ is a *K-bialgebra*.

If G is a finite group, product and coproduct of $K[G]$ are – by the functor $(\ \ )^* = Hom_K(-, K)$ – transferred to (the dual) coproduct and product on $K[G]^*$ (compare 1.13) thus making $(K[G]^*, m_G^*, \delta_G^*)$ also a K-bialgebra.

1.16 Action induced by $Aut(L : K)$

For a field extension $L : K$, put $G = Aut(L : K)$. By the action of G, L becomes a $K[G]$-module,

$$\varrho\colon K[G] \otimes_K L \to L\,, \quad g \otimes a \mapsto g(a)\,.$$

If $[L : K]$ is finite, then G is finite, say with elements $g_1, \dots, g_n$ which form a basis for $K[G]$. Choosing $p_1, \dots, p_n \in K[G]^*$ as a dual K-base for the bialgebra $K[G]^*$ (see 1.15) yields a morphism

$$\tilde{\eta}\colon K \to \sum K[G]^* \otimes_K K[G]\,, \quad 1 \mapsto \sum p_i \otimes g_i\,,$$

and leads to the K-linear map (coaction)

$$\omega\colon L \xrightarrow{\tilde{\eta}\otimes L} K[G]^* \otimes K[G] \otimes L \xrightarrow{K[G]^*\otimes\varrho} K[G]^* \otimes L\,, \quad a \mapsto \sum_i p_i \otimes g_i(a)\,.$$

Composing $\omega \otimes L$ with $K[G]^* \otimes m$ yields the map

$$\beta\colon L \otimes_K L \to K[G]^* \otimes_K L\,, \quad a \otimes b \mapsto \sum_i p_i \otimes g_i(a)b\,.$$

These constructions can be applied to prove:

Proposition 1.5. *Let $L : K$ be a field extension with $[L : K] = n$, $n \in \mathbb{N}$, and $G = Aut(L : K)$. With the notation from above, the following are equivalent:*

(a) *$L : K$ is separable and normal (Galois extension);*

(b) *for some $a \in L$, $\{g(a)\}_{g\in G}$ is a K-basis of L;*

(c) *for some $a \in L$, the map $K[G] \to L$, $g \mapsto g(a)$, is an isomorphism of $K[G]$-modules (*Normal basis theorem*);*

(d) *β is an isomorphism.*

The assertions are essentially derived from the fact that finite separable extensions can be generated by a single (primitive) element (e.g. [57, 8.1.2]). For a detailed presentation of (Hopf) Galois extension the reader may consult [77, Chapter 4].

Summarising, to any field extension $L : K$, we have attached the endofunctors $L \otimes_K -$ and the adjoint pair of functors (ϕ_L, U_L) and, as shown in Proposition 1.4, $L : K$ is separable precisely when for U_L, the canonical map

$$\Phi^U_{V,N} \colon Hom_L(V, N) \to Hom_K(U_L(V), U_L(N))$$

is a naturally split monomorphism. In the next section we shall consider functors and endofunctors for any categories and follow the above pattern to define *separable* (and *Frobenius*) functors.

2 Separable and Frobenius functors

In this section, $\mathbb{A}$ and $\mathbb{B}$ denote any categories. For basic definitions and notations we refer to the Appendix.

Definition 2.1. A functor $F \colon \mathbb{A} \to \mathbb{B}$ between categories is said to be *separable* provided, for any $A, A' \in \mathbb{A}$, the canonical map

$$\Phi^F_{A,A'} \colon Hom_{\mathbb{A}}(A, A') \to Hom_{\mathbb{B}}(F(A), F(A'))$$

is a naturally split monomorphism, that is, there is a map

$$\Psi^F_{A,A'} \colon Mor_{\mathbb{B}}(F(A), F(A')) \to Mor_{\mathbb{A}}(A, A')\,,$$

natural in A, A', with $\Psi^F_{A,A'} \circ \Phi^F_{A,A'} = 1_{\mathbb{A}}$.

Clearly, for a separable functor F, $\Phi^F_{A,A'}$ is always injective, that is, F is a faithful functor. Of course, every equivalence functor F is separable. In this context, adjoint pairs of functors are of particular interest (e.g. [12, 18, 68]).

2.1 Adjoint pairs

Let $\mathsf{L} \dashv \mathsf{R} \colon \mathbb{B} \to \mathbb{A}$ be an adjoint pair of functors with unit $\tilde{\eta} \colon 1_{\mathbb{A}} \to \mathsf{RL}$ and counit $\tilde{\varepsilon} \colon \mathsf{LR} \to 1_{\mathbb{B}}$.

(i) L is separable if and only if $\tilde{\eta}$ is a split monomorphism;

(ii) R is separable if and only if $\tilde{\varepsilon}$ is a split epimorphism.

The situation for field extensions is subsumed by (co-)monads (see [12, 2.9]). Recall that for a monad F, $\phi_F \dashv U_F$ with counit $\varepsilon_F \colon \phi_F U_F \to 1_{\mathbb{A}_F}$, and for a comonad G, $U^G \dashv \phi^G$ with unit $\eta^G \colon 1_{\mathbb{A}^G} \to \phi^G U^G$.

Definition 2.2. A monad (F, m, ι) on $\mathbb{A}$ is called *separable* if the forgetful functor $U_F\colon \mathbb{A}_F \to \mathbb{A}$ is separable.

A comonad (G, δ, ε) on $\mathbb{A}$ is called *coseparable* if the forgetful functor $U^G\colon \mathbb{A}^G \to \mathbb{A}$ is separable.

2.2 Separability for monads and comonads

(1) For a monad (F, m, ι) on $\mathbb{A}$, the following are equivalent:
   (a) (F, m, ι) is separable;
   (b) there exists a natural transformation $\delta\colon F \to FF$ with $m \cdot \delta = 1_F$ and (Frobenius condition)
$$Fm \cdot \delta F = \delta \cdot m = mF \cdot F\delta\,;$$
   (c) $\varepsilon_F\colon \phi_F U_F \to 1_{\mathbb{A}_F}$ is a split epimorphism.

(2) For a comonad (G, δ, ε) on $\mathbb{A}$, the following are equivalent:
   (a) (G, δ, ε) is coseparable;
   (b) there exists a natural transformation $m\colon GG \to G$ with $m \cdot \delta = 1_G$ and
$$mG \cdot G\delta = \delta \cdot m = Gm \cdot \delta G\,;$$
   (c) $\eta^G\colon 1_{\mathbb{A}^G} \to \phi^G U^G$ is a split monomorphism.

2.3 Separability of adjoints

Consider an adjoint pair of endofunctors $G \dashv F\colon \mathbb{A} \to \mathbb{A}$ with unit $\eta\colon 1_{\mathbb{A}} \to FG$ and counit $\varepsilon\colon GF \to 1_{\mathbb{A}}$. Assume (G, δ, ε) to be a comonad on $\mathbb{A}$ and denote by (F, m, ι) the corresponding monad (see Appendix). Then there are pairs of adjoint (free and forgetful) functors,

$$\begin{aligned} &\mathbb{A} \xrightarrow{\phi_F} \mathbb{A}_F,\ \mathbb{A}_F \xrightarrow{U_F} \mathbb{A}, &&\text{with unit } \eta_F \text{ and counit } \varepsilon_F,\\ &\mathbb{A}^G \xrightarrow{U^G} \mathbb{A},\ \mathbb{A} \xrightarrow{\phi^G} \mathbb{A}^G, &&\text{with unit } \eta^G \text{ and counit } \varepsilon^G, \end{aligned}$$

(1) ϕ^G is separable if and only if ϕ_F is separable;
(2) U^G is separable if and only if U_F is separable.

The isomorphism $\psi\colon L \to L^*$ for finite separable field extensions $L : K$ in Proposition 1.1 can be understood as natural isomorphism between the functors ϕ_L and $Hom_K(L, -)$ from $\mathbb{V}_K$ to $\mathbb{V}_L$. This is the basic property of Frobenius algebras. More generally, the role of adjoint pairs of functors for Frobenius extensions was highlighted by K. Morita in [60]. The key to this approach is the

Definition 2.3. A functor $F\colon \mathbb{A} \to \mathbb{B}$ is said to be *Frobenius* provided it has a right adjoint $G\colon \mathbb{B} \to \mathbb{A}$ which is also left adjoint to F.

Similar to separability, Frobenius (co)monads are defined by properties of the corresponding forgetful functors.

Definition 2.4. A monad (F, m, ι) on $\mathbb{A}$ is said to be a *Frobenius monad* provided the forgetful functor $U_F\colon \mathbb{A}_F \to \mathbb{A}$ is Frobenius.

A comonad (G, δ, ε) is said to be a *Frobenius comonad* provided the forgetful functor $U^G\colon \mathbb{A}^G \to \mathbb{A}$ is Frobenius (see 2.3).

As a first characterisation we note (see [54, Proposition 3.11]):

Proposition 2.1. *A monad (F, m, ι) on $\mathbb{A}$ with a right adjoint comonad (G, δ, ε) is Frobenius if and only if the functors F and G are isomorphic as left F-module functors.*

Obviously, every Frobenius monad may also be seen as a Frobenius comonad and hence it suffices to talk about Frobenius monads. We collect various characterisations of such functors (e.g. [73], [54, Theorem 3.13]).

Recall from 7.10 that, in the situation of Proposition 2.1, the categories of F-modules and of G-comodules are isomorphic.

2.4 Frobenius monads

For a monad (F, m, ι) on $\mathbb{A}$, there are equivalent:

(a) *(F, m, ι) is a Frobenius monad;*

(b) *the free functor $\phi_F\colon \mathbb{A} \to \mathbb{A}_F$ is Frobenius;*

(c) *F admits a comonad structure (F, δ, ε) and – equivalently –*

   (i) *$Fm \cdot \delta F = \delta \cdot m = mF \cdot F\delta$ (Frobenius conditions);*

   (ii) *$F \dashv F$ by unit and counit,*

$$1_{\mathbb{A}} \xrightarrow{\iota} F \xrightarrow{\delta} FF\,, \quad FF \xrightarrow{m} F \xrightarrow{\varepsilon} 1_{\mathbb{A}}\;;$$

   (iii) *an isomorphism of categories*

$$Q\colon \mathbb{A}_F \to \mathbb{A}^F \text{ with } \; U^F \cdot Q = U_F \text{ and } \; Q \cdot \phi_F \simeq \phi^F.$$

The following shows how close separable monads are to Frobenius monads.

Corollary. *Let (F, m, ι) be a monad on $\mathbb{A}$ and $\delta\colon F \to FF$ a coassociative coproduct such that (F, m, δ) satisfies the Frobenius condition (e.g. 2.2). Then:*

(1) *(F, m, ι, δ) is a separable monad if and only if $m \cdot \delta = 1_{\mathbb{A}}$.*

(2) *(F, m, ι, δ) is a Frobenius monad if and only if (F, δ) allows for a counit $\varepsilon\colon F \to 1_{\mathbb{A}}$.*

3 Separable and Frobenius algebras

We sketch the application of the categorical results from the preceding section for various special cases, beginning with the categories $\mathbb{M}_R$ of modules over a commutative ring R. Unless stated otherwise our algebras will be associative with unit.

3.1 R-algebras

A triple (A, m, ι) is called an *R-algebra* provided A is an R-module and there are R-linear maps (product and unit),

$$m\colon A \otimes_R A \to A\,, \quad \iota\colon R \to A\,,$$

satisfying the conditions to make the functor $A \otimes_R -\colon \mathbb{M}_R \to \mathbb{M}_R$ a monad. The induced module category is just the category ${}_A\mathbb{M}$ of left A-modules with the free and forgetful functors, $\phi_A\colon \mathbb{M}_R \to {}_A\mathbb{M}$ and $U_A\colon {}_A\mathbb{M} \to \mathbb{M}_R$.

The functor $- \otimes_R A$ leads to the category $\mathbb{M}_A$ of right A-modules and corresponding constructions.

The functor $A \otimes_R -$ has a right adjoint $Hom_R(A, -)\colon \mathbb{M}_R \to \mathbb{M}_R$, with unit and counit, for $X, Y \in \mathbb{M}_R$,

$$\begin{aligned} \eta\colon \quad & X \to Hom_R(A, A \otimes_R X), \quad && x \mapsto [a \mapsto a \otimes x]\,, \\ \varepsilon\colon \quad & A \otimes Hom_R(A, X) \to X, && a \otimes f \mapsto f(a)\,, \end{aligned}$$

and the bijection

$$Hom_R(A \otimes X, Y) \to Hom_R(X, Hom_R(A, Y)),\ f \mapsto [x \mapsto f(- \otimes x)]\,. \tag{1}$$

By 7.10, $Hom_R(A, -)$ is a comonad on $\mathbb{M}_R$ with coproduct and counit,

$$Hom_R(A, -) \xrightarrow{Hom(m,-)} Hom_R(A \otimes A, -) \simeq Hom_R(A, Hom_R(A, -))\,,$$
$$Hom_R(A, -) \xrightarrow{Hom(\iota,-)} 1_{\mathbb{M}_R}\,.$$

If A is finitely generated and projective as an R-module, $Hom_R(A, -) \simeq A^* \otimes_R -$. In this case, $A^* \otimes_R -$ is left and right adjoint to $A \otimes_R -$.

Comonads based on tensor functors are usually called *coalgebras*.

For an R-algebra A, the opposite algebra A^o is defined as the same R-module with opposite multiplication. Then the (A, A)-bimodules can be considered as (left) modules over the algebra $A^e := A \otimes_R A^o$. In particular, A is a left A^e-module by the action

$$\mu_A\colon (A \otimes A^o) \otimes A \to A\,, \quad a \otimes b \otimes c \mapsto acb\,. \tag{2}$$

and its endomorphism ring $\mathrm{End}_{A^e}(A)$ is just its center $Z(A)$.

Now Definition 2.1 yields:

Definition 3.1. An R-algebra (A, m, ι) is called *separable* if the induced monad on $\mathbb{M}_R$ is separable, that is, $U_A\colon \mathbb{M}_A \to \mathbb{M}_R$ is a separable functor, which means that, for $M, N \in {}_A\mathbb{M}$, the canonical map

$$\Phi^A_{M,N}\colon Hom_A(M, N) \to Hom_R(U_A(M), U_A(N))$$

is a (naturally) split monomorphism.

This generalises the characterisation of separable field extensions in 1.14 and we derive from 2.2:

3.2 Separable algebras

For an R-algebra A, the following are equivalent:

(a) *A is a separable algebra;*

(b) *$m\colon A \otimes_R A \to A$ splits as an $A \otimes_R A^o$-module morphism (by some $\delta\colon A \to A \otimes_R A$);*

(c) *there exists $e \in A \otimes_R A^o$ with $(a \otimes 1)e = (1 \otimes a)e$ for $a \in A$ and $m(e) = 1$ (choose $e = \delta(1)$, separability idempotent);*

(d) *every A-epimorphism which splits as R-module map splits as an A-module map (A is (A, R)-semisimple).*

To show (b)⇒(a), the splitting of $Hom_A(M, N) \to Hom_R(M, N)$, for $M, N \in {}_A\mathbb{M}$, is given by sending any R-morphism $g\colon M \to N$ to the composite

$$M \xrightarrow{\iota\otimes M} A \otimes M \xrightarrow{\delta\otimes M} A \otimes A \otimes M \xrightarrow{A\otimes\varrho_M} A \otimes M \xrightarrow{A\otimes g} A \otimes N \xrightarrow{\varrho_N} N.$$

Characterisation (c) is possible, since the A^e-linear map δ is uniquely determined by the image of 1_A yielding the *separability idempotent* $\delta(1_A)$ which is often used to prove properties of these algebras.

The condition in (b), reducing the property to the splitting of a single linear map, was used by M. Auslander and O. Goldman in their paper [5] and before a special case of this was considered by G. Azumaya in [6].

Definition 3.2. A separable R-algebra (A, m, ι) is said to be *strongly separable* if the separability idempotent $e \in A \otimes_R A^o$ (in Theorem 3.2(c)) is symmetric, that is,

$$e = \sum_i e_i \otimes f_i = \sum_i f_i \otimes e_i\,.$$

Such algebras were considered by A. Hattori in [32] and more observations on these were made, among others, by L. Kadison and A. Stolin [37], and M. Aguiar in [4].

One question of interest is which of the properties of field extension can be obtained for algebras A over a field K. For a finite dimensional A, the K-endomorphism ring $\mathrm{End}_K(A)$ is a matrix ring, canonically isomorphic to $A \otimes_K A^*$, and so we get the K-linear map

$$\mathrm{tr}\colon A \xrightarrow{\lambda} \mathrm{End}_K(A) \simeq A \otimes A^* \xrightarrow{ev} K\,, \quad a \mapsto Tr(\lambda_a)\,, \tag{3}$$

where ev denotes the evaluation map, and $\mathrm{End}_K(A) \to K$ just gives the trace map Tr. For $a, b, c \in A$, we have

$$\mathrm{tr}(ab) = \mathrm{tr}(ba)\,, \quad \mathrm{tr}(a(bc)) = \mathrm{tr}((ab)c)\,,$$

that is, tr is a symmetric and balanced linear form. It is called *non-degenerate* provided the K-linear map

$$A \to A^*,\ a \mapsto \mathrm{tr}(a-)\,,$$

is a K-linear isomorphism.

We note that this construction can also be made for algebras A over commutative rings R provided A is finitely generated and projective as R-module.

3.3 Separable algebras over fields

Let A be a finite dimensional algebra A over a field K.
(1) The following are equivalent:

- (a) *A is a separable K-algebra (Definition 3.1);*
- (b) *for every field extension $L : K$, $A \otimes_K L$ is a (finite) direct product of simple algebras.*
- (c) *for any field extension $L : K$, $rad(A \otimes_K L) = 0$:*
- (d) *$A \otimes_K A^o$ is a direct product of simple algebras.*

(2) Furthermore, the following are equivalent:

- (a) *A is a strongly separable K-algebra (Definition 3.2);*
- (b) *$tr\colon A \to K$ is a non-degenerate linear form.*

The equivalence of (b) and (c) in (1) is due to the fact that for any finite dimensional algebra the prime radical is zero if and only if the algebra is semisimple. For the equivalences stated in (2) we refer to [4, Theorem 3.1]. Notice that here it is not enough to require tr to be non-zero (as it is for field extensions).

Definition 3.3. *A separable R-algebra A with center $Z(A) = R$ is called* central separable *or* Azumaya algebra.

The class of these structures can be characterised in the following way.

3.4 Azumaya algebras

For a central R-algebra (A, m, ι), there are equivalent:

(a) *A is a separable R-algebra;*

(b) *there is an A^e-linear map $\delta\colon A \to A \otimes_R A$ with $m \cdot \delta = 1_A$;*

(c) *A induces an equivalence of categories,*

$$A \otimes_R -\colon \mathbb{M}_R \to {}_{A^e}\mathbb{M}, \quad X \mapsto (A \otimes_R X, m \otimes X)\,;$$

(d) *A as a left A^e-module is a generator in ${}_{A^e}\mathbb{M}$.*

We note that in any full module category over a ring, generators are finitely generated and projective over their endomorphism rings (e.g. [88, 18.8]). Thus an equivalence as given in (c) always forces this kind of finiteness condition on the R-module structure of A.

On the other hand, there are simple algebras over rings (and fields) which are not finitely generated over their centers (which are fields). To include this type of algebras in our theory we proceed in the following way.

The basic idea of the above characterisations is to relate internal properties of an algebra A (as given in (a)) with properties in a suitable category (as in (e)). The category ${}_{A^e}\mathbb{M}$ has coproducts and cokernels, products and kernels (Grothendieck category). We restrict our considerations to a (smallest) subcategory with similar properties which contains the A^e-module A: So we form a full category $\sigma_{A^e}[A]$ by taking as objects all direct sums of copies of A, all homomorphic images and submodules of the resulting modules. This gives us again a Grothendieck category (see [87]).

Definition 3.4. A central R-algebra (A, m, ι) is called an *Azumaya ring* provided the A^e-module A is a projective generator in $\sigma_{A^e}[A]$.

These algebras were also investigated by J.P. Delale in [21] under the name *algèbres affines*, by G. Azumaya in [7] under the name *separable rings*, and D.G. Burkholder called them *Azumaya rings* in [16]. The notion was extended to non-associative algebras in [84]. Any simple algebra A is a simple A^e-module and hence every module in $\sigma_{A^e}[A]$ is semisimple, in fact a direct sum of copies of A, and hence A is a projective generator in $\sigma_{A^e}[A]$, i.e., A is an Azumaya ring.

Generators in $\sigma_{A^e}[A]$ are flat as modules over their endomorphism rings and we get the following characterisations (see [87, 26.4]).

3.5 Azumaya rings

For a central R-algebra (A, m, ι), the following are equivalent:

(a) *A is an Azumaya ring;*

(b) *A induces an equivalence of categories,*

$$A \otimes_R -: \mathbb{M}_R \to \sigma_{A^e}[A], \ X \mapsto (A \otimes_R X, m_a \otimes X)\ ;$$

(c) *A is a generator in $\sigma_{A^e}[A]$ and A is faithfully flat over R;*

(d) *$Hom_{A^e}(A,-) : \sigma_{A^e}[A] \to \mathbb{M}_R$ is an equivalence of categories.*

As easily seen, a central algebra A is an Azumaya algebra if and only if A is an Azumaya ring and A is finitely generated as an R-module. In this case one has $\sigma_{A^e}[A] = {}_{A^e}\mathbb{M}$.

So far we have considered algebras with unit. The missing of an identity element changes the picture in some aspects.

3.6 Algebras without units

For a non-unital R-algebra (A, m), $A \otimes_R -$ is no longer a monad, A^e may have no unit and hence need neither be projective nor a generator in ${}_{A^e}\mathbb{M}$. Furthermore, $\mathrm{End}_{A^e}(A)$ is no longer the center of the algebra A, it is called the *centroid* $C(A)$ of A (e.g. [87, 2.7]). If $A = A^2$, then $C(A)$ is a commutative R-algebra and A is a $C(A)$-algebra which is called *central* if $C(A) \simeq R$. The definition of Azumaya rings also hold for non-unital algebras. However, their characterisations considered above are to be modified (see [87, Chapter 7]). Non-unital separable algebras may be defined by the (A, A)-bimodule splitting of $m\colon A^e \to A$. They are investigated, for example, by Taylor in [76] to study a *bigger Brauer group*.

Most of the preceding results in this section depend heavily on the commutativity of the base ring (or field) R, which allows the twist map $X \otimes_R Y \to Y \otimes_R X$, $x \otimes y \mapsto y \otimes x$. This is, for example, needed to define multiplication on the tensor product $A \otimes_R A^o$. Yet some constructions are still possible in more general situations.

3.7 Non-commutative base ring

Let (A, m, ι) be a ring ($\mathbb{Z}$-algebra). For a ring extension $B \to A$, $A \otimes_B -\colon {}_B\mathbb{M} \to {}_B\mathbb{M}$ defines a monad on ${}_B\mathbb{M}$, and if this is separable, the extension is said to be *separable*. As for algebras, this is the case if $m\colon A \otimes_B A \to A$ splits as an (A, A)-bimodule map. The investigation of this case was initiated by K. Hirata and S. Sugano in [34]. Several properties of separable algebras are maintained but here $A \otimes_B A$ need not have a ring structure and hence the results from subsection 3.2 need modification.

For example, for the separable extension, A need not be a projective generator for the (A, A)-bimodules. To replace this, K. Hirata in [33], suggested a stronger condition for separability (and K. Sugano coined its name [74]):

Definition 3.5. A ring extension $B \to A$ is called *H-separable* if, for some $n \in \mathbb{N}$, $A \otimes_B A$ is a direct summand of A^n as an (A, A)-submodule.

As observed in [33, Theorem 2.2], any H-separable extension is a separable extension. These structures are investigated in a series of papers by K. Hirata, K. Sugano, T. Nakamoto, Y. Kurata, S. Morimoto, S., F. Kasch and B. Pareigis [33, 39, 45, 61, 74], and others.

3.8 Separability and modules

The notion of separability applies to any functors. In particular, one may ask when functors related to bimodules have this property. Early papers considering this question were Sugano [75] and Cunningham [19].

Definition 3.6. A bimodule ${}_AM_B$ over any rings A, B is said to be *separable* provided the functor

$$Hom_A(M, -) \colon {}_A\mathbb{M} \to {}_B\mathbb{M} , \quad X \mapsto Hom_A(M, X) , \tag{4}$$

is a separable functor.

Since $M \otimes_B -$ is left adjoint to $Hom_A(M, -)$, it follows from 2.1 that the latter is separable if and only if, for any $X \in {}_A\mathbb{M}$, the evaluation map

$$ev_X \colon M \otimes_B Hom_A(M, X) \to X , \quad m \otimes f \mapsto f(m) ,$$

is a naturally split epimorphism.

If M is finitely generated and projective as an A-module, the condition can be reduced to require that the evaluation

$$ev_A \colon M \otimes_B Hom_A(M, A) \to A , \quad m \otimes f \mapsto f(m) , \tag{5}$$

is naturally split as an (A, A)-bimodule map. This situation was investigated in [75] and further results in this direction are given in [19] and [86].

Remark. The functor $Hom_A(M, -)$ in (4) can be restricted to the category $\sigma[{}_AM]$, the full subcategory of ${}_A\mathbb{M}$ subgenerated by ${}_AM$ (e.g. [88]). Then it still has the functor $M \otimes_B - \colon {}_B\mathbb{M} \to \sigma[M]$ as a left adjoint and the results from Section 2 apply. However, separability of $Hom_A(M, -)$ does not imply that ev_A from 5 is surjective (unless $A \in \sigma[{}_AM]$). Compare also Theorem 3.5.

This kind of studies were also pursued by M. Sato in [70] where some properties of tilting and static modules are anticipated (e.g. [89, 90]).

For non-associative R-algebras (A, m), the endofunctor $A \otimes_R -$ is no monad and to study the relevance of separability in this case, one looks at the structure of closely related associative unital algebras.

3.9 Multiplication algebra

For any R-algebra A, not necessarily associative nor with unit, the left and right multiplications with elements $a \in A$ yield R-linear endomorphisms of A,

$$\lambda_a : A \to A,\ x \mapsto ax\,, \quad \rho_a : A \to A,\ x \mapsto xa\,,$$

Denote by $M(A)$ the subalgebra of $\mathrm{End}_R(A)$ generated by all $\{\lambda_a, \rho_a \,|\, a \in A\}$ and the identity map of A. $M(A)$ is called the *multiplication algebra* of A and since A is a module over $\mathrm{End}_R(A)$, it is also a module over $M(A)$, in fact an $(M(A), R)$-bimodule. The $M(A)$-submodules are just the two-sided ideals of A and hence the $M(A)$-module structure of A reflects its ring theoretic properties.

In case A is an associative unital algebra, there is a surjective algebra homomorphism

$$A^e = A \otimes A^o \to M(A)\,, \quad a \otimes b \mapsto \lambda_a \rho_b\,,$$

and hence the A^e-module structure and the $M(A)$-module structure of A coincide. In fact, the attached categories $\sigma_{A^e}[A]$ and $\sigma_{M(A)}[A]$ can be identified and large parts of our results for separability and generating properties as A^e-modules can be formulated in terms of $M(A)$-modules. For example, A is an Azumaya algebra if and only if it is a generator in ${}_{M(A)}\mathbb{M}$ since, in this case, $A^e \simeq M(A) \simeq \mathrm{End}_R(A)$.

3.10 Non-associative algebras

The description of properties of non-associative algebras A by their multiplication algebras $M(A)$ was already considered by A.A. Albert and N. Jacobson (e.g. [36]). They observed that a finite dimensional algebra A over a field K is separable if and only if $M(A)$ is a separable (associative) K-algebra.

Considering the category $\sigma_{M(A)}[A]$ introduced in 3.9, one obtains a rich theory for non-associative algebras over rings without a priori finiteness restrictions. For example, characterisations of nonassociative Azumaya rings are obtained by simply replacing in 3.5 the algebra A^e by $M(A)$. This is elaborated in [87]. In this context, the algebra A^e, known as *enveloping algebra* in the associative case, is not relevant. Instead, for certain classes of non-associative algebras, e.g. alternative, Jordan or Lie algebras, the corresponding (associative) *enveloping algebras* can enter the picture (e.g. [10, 11, 28, 36, 87, 29.10]). For the study of regularity and radicals for non-associative algebras we refer to [85] and [82].

In 3.3, strong separability of associative algebras is characterised by a non-degenerate associative linear form $A \otimes_R A \to R$. This can also be achieved for some non-associative algebras, for example, certain *composition algebras* (e.g. [83]). For Lie algebras, separability is given by nondegeneracy of the *Killing form* (e.g. [20, 65]). In [80], a *Killing form* is also defined for Hopf algebras.

For associative algebras over a commutative ring R, Definition 2.4 yields:

Definition 3.7. An R-algebra (A, m, ι) is said to be a *Frobenius algebra* provided the forgetful functor $U_A : {}_A\mathbb{M} \to \mathbb{M}_R$ is Frobenius.

The categorical characterisations of this notion in 2.4 read now:

3.11 Frobenius algebras

For an associative R-algebra (A, m, ι), there are equivalent:

(a) *(A, m, ι) is a Frobenius algebra;*

(b) *the free functor $\phi_A : \mathbb{M}_R \to {}_A\mathbb{M}$ is Frobenius;*

(c) *A admits a comonad structure (A, δ, ε) and – equivalently –*

  (i) *(A, m, δ) satisfies the Frobenius conditions, that is,*

$$Am \cdot \delta A = \delta \cdot m = mA \cdot A\delta \; ; \text{ or}$$

  (ii) *$A \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R$ is adjoint to itself with unit and counit,*

$$1_{\mathbb{A}} \xrightarrow{\iota} A \xrightarrow{\delta} A \otimes_R A \, , \quad A \otimes_R A \xrightarrow{m} A \xrightarrow{\varepsilon} 1_{\mathbb{A}} \; ;$$

(d) *$A \otimes_R - \simeq Hom_R(A, -)$ as left A-module functors on $\mathbb{M}_R$;*

(e) *A is finitely generated and projective as an R-module and $A \simeq A^*$ as left (or right) A-modules.*

The isomorphism in (d) means that $A \otimes_R -$ preserves direct products and monomorphisms (as right adjoints do), that is, A_R has to be flat and finitely presented, hence finitely generated and projective as an R-module. In this case $Hom_R(A, -) \simeq A^* \otimes_R -$.

For an R-module A with an algebra and a coalgebra structure, one may attach the categories of left A-modules ${}_A\mathbb{M}$ and left comodules ${}^A\mathbb{M}$. Then A yields a Frobenius algebra if and only if there is an isomorphism of categories (see 2.4, [54, Theorem 3.13])

$$Q : {}_A\mathbb{M} \to {}^A\mathbb{M} \quad \text{with} \quad U^A \cdot Q = U_A \quad \text{and} \quad Q \cdot \phi_A \simeq \phi^A .$$

By Proposition 1.1, finite separable field extensions $L : K$ allow for an L-isomorphism $L \simeq L^*$. This may have been one of the motivations for F. Frobenius to investigate (in [30], 1903) finite dimensional algebras A with the corresponding property. In [26] (1955), Eilenberg and Nakayama observed that the notion makes sense for algebras A over commutative rings R, provided A is finitely generated and projective as an R-module. These turned out to be of considerable interest in various areas of mathematics and theoretical physics.

Corollary. *Let A be a finitely generated and projective R-module allowing for an algebra structure (A, m, ι) and a coassociative coproduct $\delta : A \to A \otimes_R A$ such that (A, m, δ) satisfies the Frobenius conditions. Then:*

(1) *A is a separable algebra if and only if $m \cdot \delta = 1_A$.*

(2) *A is a Frobenius algebra if and only if (A, δ) allows for a counit $\varepsilon\colon A \to R$.*

Definition 3.8 (Frobenius extensions). For ring extensions $B \to A$ with non-commutative rings A, B, the functor $A \otimes_B -\colon {}_B\mathbb{M} \to {}_B\mathbb{M}$ allows for a monad structure (see 3.7) and if this is Frobenius, the extension is called *Frobenius extension*, that is, the forgetful functor $U_A\colon {}_A\mathbb{M} \to {}_B\mathbb{M}$ is Frobenius.

The theory of such extensions was initiated by F. Kasch [38], T. Nakayama and T. Tsuku [62], K. Morita [60] and the literature around it is abundant.

4 Coseparable and Frobenius coalgebras

The categorical setting readily provides properties of coseparable comonads. In classical algebra, mainly comonads based on a tensor functor are considered and then are called *coalgebras*. We sketch the resulting framework for the category $\mathbb{M}_R$ of modules over a commutative ring R.

4.1 Coalgebras

A triple (C, δ, ε) is called an *R-coalgebra* provided C is an R-module and there are R-linear maps

$$\delta\colon C \to C \otimes_R C\,, \quad \varepsilon\colon C \to R\,,$$

satisfying the conditions to make the functor $C \otimes_R -\colon \mathbb{M}_R \to \mathbb{M}_R$ a comonad on $\mathbb{M}_R$ (see 7.6).

Notice that every free R-module V, with basis x_i, $i \in I$ any set, has a comodule structure by defining

$$\delta\colon x_i \mapsto x_i \otimes x_i\,, \quad \varepsilon\colon x_i \mapsto 1\,, \quad \text{for } i \in I\,,$$

and extending this linearly to all of V.

From 7.6 we derive the notion of *C-comodules* obtaining the category of left comodules ${}^C\mathbb{M}$ (in which monomorphisms need not be injective and morphism need not have kernels). The morphism sets in ${}^C\mathbb{M}$ are denoted by Hom^C. We have the adjoint pair of forgetful and free functors,

$$\begin{aligned} U^C\colon {}^C\mathbb{M} \to \mathbb{M}_R, &\quad (M, \omega) \mapsto M\,, \\ C \otimes_R -\colon \mathbb{M}_R \to {}^C\mathbb{M}, &\quad X \mapsto (C \otimes_R X, \delta \otimes X)\,, \end{aligned}$$

by the bijection

$$\begin{aligned} Hom^C(M, C \otimes_R X) \to &\quad Hom_R(M, X)\,, \\ M \xrightarrow{h} C \otimes_R X \ \mapsto &\quad M \xrightarrow{h} C \otimes_R X \xrightarrow{\varepsilon \otimes X} X\,, \end{aligned}$$

leading, in particular, to the isomorphisms

$$\mathrm{End}^C(C) \simeq Hom_R(C, R) = C^* ,$$
$$Hom^C(M, C) \simeq Hom_R(M, R) = M^* .$$

Of course, for a coalgebra (C, δ, ε), the setting of 7.6 also applies to the endofunctor $- \otimes_R C\colon \mathbb{M}_R \to \mathbb{M}_R$ leading to the category of *right C-comodules* $\mathbb{M}^C$. Clearly, C itself is a left as well as a right C-comodule.

The functor $C \otimes_R -$ has as right adjoint the functor $Hom_R(C, -)\colon \mathbb{M}_R \to \mathbb{M}_R$ and the coalgebra structure on $C \otimes_R -$ induces a monad structure on $Hom_R(C, -)$. The modules for this monad are also called *C-contramodules* (e.g,[12, 27, 93]). Following Definition 2.2 we have:

Definition 4.1. An R-coalgebra (C, δ, ε) is said to be *coseparable* if the forgetful functor $U^C\colon {}^C\mathbb{M} \to \mathbb{M}_R$ is separable.

From 2.2 we derive (see also [15, 3.29]):

Proposition 4.1 (Coseparable coalgebras). *For an R-coalgebra (C, δ, ε), the following are equivalent:*

(a) *C is a coseparable R-coalgebra;*
(b) *$\delta\colon C \to C \otimes C$ splits as a left and right C-comodule map (by some $m\colon C \otimes_R C \to C$);*
(c) *every C-monomorphism which splits as an R-module map splits as a C-comodule map (C is (C, R)-semisimple).*

As an application of 2.3 we obtain:

Proposition 4.2. *An R-coalgebra (C, δ, ε) is coseparable if and only if the induced monad on $Hom_R(C, -)$ is separable.*

If C is finitely generated and projective as an R-module, this means that C^* (with the convolution product) is a separable R-algebra.

For the investigation of R-coalgebras we have heavily used that the tensor product of two R-modules is again an R-module, that is, $\mathbb{M}_R$ allows for a tensor product (monoidal category). For a non-commutative ring A, endofunctors for the category of left A-modules can be provided by (A, A)-bimodules M, that is,

$$M \otimes_A -\colon {}_A\mathbb{M} \to {}_A\mathbb{M} , \quad X \mapsto M \otimes_A X .$$

This is the basis for our next definition.

Definition 4.2 (Corings). Let A be a ring. A triple (C, δ, ε) is called an *A-coring* provided C is an (A, A)-bimodule and there are (A, A)-linear maps

$$\delta\colon C \to C \otimes_A C , \quad \varepsilon\colon C \to A ,$$

satisfying the conditions to make the functor $C \otimes_A -\colon {}_A\mathbb{M} \to {}_A\mathbb{M}$ a comonad on ${}_A\mathbb{M}$ (see 7.6).

Left comodules over (C, δ, ε), the category of left comodules ${}^C_A\mathbb{M}$, the free and forgetful functors $\phi^C\colon {}_A\mathbb{M} \to {}^C_A\mathbb{M}$, $U^C\colon {}^C_A\mathbb{M} \to {}_A\mathbb{M}$ are defined by the corresponding definitions for comonads. In particular, *coseparable corings* are defined as coseparable comonads (see Definition 2.2). The results for coalgebras – with obvious modifications – apply widely to corings and this is outlined in detail in [15].

An interesting class of corings is obtained by tensoring an R-algebra A with an R-coalgebra C, R a commutative ring:

4.2 Entwining algebras and coalgebras

Consider an R-algebra (A, m, ι), an R-coalgebra (C, δ, ε) and a mixed entwining between the functors $A \otimes_R -$ and $C \otimes_R -$, that is, an R-linear map

$$\lambda\colon A \otimes_R C \to C \otimes_R A$$

with commutative diagrams as in 7.7. This makes $C \otimes_R A$ a left A-module by

$$A \otimes_R C \otimes_R A \xrightarrow{\lambda\otimes C} C \otimes_R A \otimes_R A \xrightarrow{C\otimes m} C \otimes_R A \,.$$

Together with the canonical right A-module structure, $C \otimes_R A$ is an (A, A)-bimodule and the coproduct and counit,

$$C \otimes_R A \xrightarrow{\delta\otimes A} C \otimes_R C \otimes_R A \simeq (C \otimes_R A) \otimes_A (C \otimes_R A)\,, \quad C \otimes_R A \xrightarrow{\varepsilon\otimes A} A\,,$$

create a comonad on ${}_A\mathbb{M}$, that is, an A-coring (see 4.2).

We end this section with a further view on group actions. Recall that in 1.16, a $K[G]^*$-comodule structure on L is considered. This is a special case of a Hopf algebra H coacting on an algebra, namely an

4.3 *H*-comodule algebra

Let H be an R-Hopf algebra and (A, m, ι) an R-algebra with coaction $\omega\colon A \to A \otimes_R H$. A is said to be an *H-comodule algebra* provided m and ι are H-colinear. Then the *coinvariants of H in A*, defined as

$$A^{coH} = \{a \in A \mid \omega(a) = a \otimes 1_H\}\,,$$

form a subalgebra of A. The extension $A^{coH} \subset A$ is called (right) *H-Galois* whenever the map

$$\beta\colon A \otimes_{A^{coH}} A \to A \otimes_R H\,, \quad (a \otimes 1_H)\,\omega(b)\,,$$

is bijective. These structures are addressed, for example, in [57], [77, Chapter 4], and reconsidered for corings, e.g., in [15, 91]. A comprehensive survey is given in [58].

5 Application to semirings

In the last decades, semirings have turned out to be of considerable interest in various fields of pure and applied mathematics. For example, bialgebras over semirings offer an advantageous framework for automata and formal language theory (see [96]). For a survey and an introduction to this field of research the reader is referred to the book of J.S. Golan [31]. Aspects of linear algebra over semirings are nicely presented in D. Wilding's thesis [81]. We recall the basic notions in a form suitable for our setting.

A commutative monoid $(M, +)$ is a set M together with an associative and commutative composition (addition)

$$+ \colon M \times M \to M, \ (m, m') \mapsto m + m' ,$$

and a neutral element $0 \in M$, that is, $0 + m = m$, for all $m \in M$. The prototype for this structure are the non-negative integers, denoted by $\mathbb{N}_0$.

Morphisms between two (commutative) monoids are maps $f \colon M \to N$ respecting the addition and the neutral element. These data define the category $\textsc{Mon}$ of commutative monoids.

A *tensor product* between two objects $M, N \in \textsc{Mon}$ is defined as a bilinear map $\tau \colon M \times N \to M \otimes N$, for some $M \otimes N \in \textsc{Mon}$, with the universal property: For any $L \in \textsc{Mon}$ and bilinear map $\beta \colon M \times N \to L$, there is a unique monoid map $\widetilde{\beta} \colon M \otimes N \to L$ with commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "\tau"'] & L \\ M \otimes N \arrow[ur, "\widetilde{\beta}"'] & \end{tikzcd} .$$

The existence of such a tensor product is shown by taking the free commutative monoid F generated by the set $M \times N$, and then the quotient monoid by the congruence relation on F generated by all pairs of the form

$$((m + m', n); (m, n) + (m', n)); \ ((m, n + n'); (m, n) + (m, n')) ,$$

with $m, m' \in M$, $n.n' \in N$. (e.g. [2, 8, 40, 67]).

Any object $R \in \textsc{Mon}$ defines a functor $R \otimes - \colon \textsc{Mon} \to \textsc{Mon}, \ X \mapsto R \otimes X$, and it is called a *semiring* if this functor allows for a monad structure, that is, an (associative) product $m \colon R \otimes R \to R$ and a unit $\iota \colon \mathbb{N}_0 \to R$. A *morphism* $f \colon R \to S$ of semirings is a map respecting the defining operations.

A *left R-semimodule* is a module for the functor $R \otimes - \colon \textsc{Mon} \to \textsc{Mon}$, that is, an $M \in \textsc{Mon}$ together with a morphism $\rho \colon R \otimes M \to M$ in $\textsc{Mon}$, and *morphism of left R-semimodules* $M \to N$ are the module morphisms for $R \otimes -$, that is, monoid morphisms respecting the scalar multiplication, we denote them by $Hom_R(M, N)$. These data define the category ${}_R\textsc{Mon}$ of left R-semimodules.

Right R-semimodules and their category MON_R are derived from the functor $-\otimes R$: $\text{MON} \to \text{MON}$.

Given two semirings R and S, a commutative monoid U with a left R-semimodule and a right S-module structure is said to be an (R, S)-*bisemimodule* provided these actions commute.

The tensor product between a right R-semimodule $\varrho\colon M \otimes R \to M$ and a left R-semimodule $\varrho'\colon R \otimes N \to N$ is defined by the coequaliser in MON,

$$M \otimes R \otimes N \underset{M\otimes\varrho'}{\overset{\varrho\otimes N}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_R N.$$

If R is a commutative semiring, left R-semimodules M may be considered as right R-modules by putting $rm = mr$, for $m \in M$, $r \in R$. Then $M \otimes_R N$ obtains a canonical structure as an R-semimodule.

For an (R, S)-bisemimodule U, the functors

$$U \otimes_S -\colon {}_S\text{MON} \to {}_R\text{MON} \quad \text{and} \quad Hom_R(U, -)\colon {}_R\text{MON} \to {}_S\text{MON}$$

are adjoint by the (canonical) bijection, for $M \in {}_S\text{MON}$, $N \in {}_R\text{MON}$,

$$Hom_R(U \otimes_S M, N) \to Hom_S(M, Hom_R(U, N)),\ \ h \mapsto [m \mapsto h(- \otimes m)]\ .$$

Let R be any semiring. An (R, R)-bisemimodule A yields an endofunctor $A \otimes_R -\colon {}_R\text{MON} \to {}_R\text{MON}$, and A is said to be an R-*semiring* provided this functor allows for a monad structure. Similarly, an (R, R)-bisemimodule C is called an R-*semicoring* if this functor allows for a comonad structure.

In case R is a commutative semiring, left (right) R-semimodules are considered as R-bimodules (as mentioned above), and the R-semirings are called R-*semialgebras* and R-semicorings are named R-*coalgebras*.

The categorical background leads to the definitions and properties of the corresponding categories of modules and comodules, respectively, and the appropriate free and forgetful functors.

Definition 5.1. Let R be any semiring. An R-semiring A is called *separable* (*Frobenius)* if the forgetful functor $U_A\colon {}_A\text{MON} \to {}_R\text{MON}$ is separable (Frobenius). An R-semicoring C is called *coseparable* (*Frobenius*) if the forgetful functor $U^C\colon {}^C\text{MON} \to {}_R\text{MON}$ is separable (Frobenius).

Properties of theses structures can be derived from the results in Section 2. Further investigation on these notions may be of interest. Separable and central separable cancellative R-semialgebras (R a commutative semiring) are considered by R.P. Deore e.a. in [23, 24].

In the setting considered above, the definition of bimonads and Hopf monads in [50] provides the definition of *bisemialgebras* and *Hopf semialgebras*. This is worked out by J. Abuhlail and N. Al-Sulaiman in [3] where also separability for Hopf semialgebras is discussed. Semialgebras, semi-coalgebras and bi-semialgebras are used by

J. Worthington as tools for automata theory in [96]. The Sweedler dual of a bialgebra over semirings is considered by G.H.E. Duchamp and C. Tollu in [25]. An extension of the Myhill-Nerode theorem to base semirings is subject of [49].

5.1 Near-semirings

For a set S, a quadruple $(S, +, \cdot, 0)$ is called a *(right) near-semiring* (also *semi-nearring*) if

(i) $(S, +, 0)$ is a monoid, $(S, \cdot)$ is a semigroup,

(ii) $0 \cdot a = 0$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

A near-semiring $(S, +, \cdot 0)$ is a *nearring* if $(S, +, 0)$ is a (not necessarily commutative) group, and it is a *semiring*, if $(S, +, 0)$ is commutative and, in addition, $a \cdot 0 = 0$ and $c(a + b) = ca + cb$ for all $a, b, c \in S$.

The interest in these structures comes from the following observation: for any monoid $(N, +, 0)$, the maps from N to N, write $Map(N, N)$, form a near-semiring with respect to pointwise addition and composition of mappings. If $(N, +, 0)$ is a group, then $Map(N, N)$ is a nearring. Note that the terminology may differ in the literature.

These notions are also of interest in automata theory and computer science. To get an adjoint pair of functors between categories of interest observe that for an object A in any category $\mathbb{A}$, there is a functor $Mor_{\mathbb{A}}(A, -)\colon \mathbb{A} \to \textsc{Set}$. A functor $T\colon \textsc{Set} \to \mathbb{A}$ that is left adjoint to it exists, provided for every family of copies of A there is a coproduct in $\mathbb{A}$. Based on this observation, a generalisation of nearrings and related tensor products are considered in [29]. Representations of near-semirings are investigated in [43].

6 Application to *S*-acts

In the preceding section we considered semiring actions on commutative monoids. More generally, one may also study the action of monoids on any set. This is the appropriate setting for the theory of *automata* and we recall the basic definitions (e.g. [41]).

Let S be any monoid, that is, a set S with associative product $m\colon S \times S \to S$ and unit element 1_S. Then the endofunctor

$$S \times -\colon \textsc{Set} \to \textsc{Set}\,, \quad X \mapsto S \times X\,,$$

is a monad on Set. The (Eilenberg-Moore) modules of this functor (see 7.5) are called *(left) S-acts*. These are sets A with an associative and unital S-action

$$\varrho\colon S \times A \to A,\ (s, a) \mapsto sa\,.$$

We denote the category of S-acts by ${}_S\textsc{Set}$.

Induced by $S \times -$, we get the free and forgetful functors

$$\phi_S\colon \mathrm{Set} \to {}_S\mathrm{Set}, \quad X \mapsto (S \times X, m_X)\,,$$
$$U_S\colon {}_S\mathrm{Set} \to \mathrm{Set}, \quad (X, \varrho) \mapsto X\,.$$

Right S-acts are determined by the endofunctor $- \times S\colon \mathrm{Set} \to \mathrm{Set}$ and yield the category Set_S. For monoids S, T, (S, T)-*biacts* are given by commuting left S- and right T-actions.

In the literature, S-acts also show up under the name *S-sets, S-polygons, transition system, S-automata*, indicating the area where they are of interest.

Given any $(A, \varrho) \in \mathrm{Set}_S$, $(B, \varrho') \in {}_S\mathrm{Set}$, their *tensor product* is defined as the coequaliser in Set,

$$\begin{tikzcd} A \times S \times B \arrow[r, shift left, "\varrho\times B"] \arrow[r, shift right, "A\times\varrho'"'] & A \times B \arrow[r, "\tau"] & A \otimes_S B, \end{tikzcd}$$

and thus is characterised by the universal property: for any set Y and S-balanced map $\beta\colon A \times B \to Y$ (i.e., $\beta(as, b) = \beta(a, sb)$, for $a \in A$, $b \in B$, $s \in S$), there is a unique map $\widetilde{\beta}\colon A \otimes_S B \to Y$ with commutative diagram

$$\begin{tikzcd} A \times B \arrow[r, "\beta"] \arrow[d, "\tau"'] & Y \\ A \otimes_S B \arrow[ur, "\widetilde{\beta}"'] & \end{tikzcd} .$$

Evidently, the formalism and the basic properties for this tensor product are the same as for the tensor product for modules and semimodules.

Given two monoids S and T, any (S, T)-biact A induces the adjoint pair of functors

$$A \otimes_T -\colon {}_T\mathrm{Set} \to {}_S\mathrm{Set}, \quad X \mapsto A \otimes_T X\,,$$
$$Hom_S(A, -)\colon S_{\mathrm{Set}} \to {}_T\mathrm{Set}, \quad Y \mapsto Hom_S(A, Y)\,,$$

with the canonical bijection

$$Hom_S(A \otimes_T X, Y) \to Hom_T(X, Hom_S(A, Y)),\ \ h \mapsto [x \mapsto h(- \otimes x)]\,.$$

Remark. For any monoid $(S, m, 1_S)$, $S\times-$ is a monad with a coproduct given by $d\colon S \to S\times S,\ s \mapsto (s, s)$. An easy argument shows that (S, m, d) does not satisfy the Frobenius conditions unless S consists only of one element. Furthermore, $m \cdot d = 1_S$ if and only if all elements of S are idempotent, i.e., S is a Boolean monoid.

On the other hand, $d\colon S \to S \times S$ respects the product and thus m and d are compatible in the sense of 7.8. Also, the map $\varepsilon\colon S \to 1_{\mathrm{Set}},\ s \mapsto [\omega]$, where $[\omega]$ is a singleton, makes (S, d, ε) a comonad such that $(S, m, 1_S, d, \varepsilon)$ becomes a bimonad. This is a Hopf monad if and only if the monoid S is a group (e.g. [92, 5.19]).

For the use of distributive laws in the theory of automata we refer to [17].

7 Appendix: Categorical background

For the convenience of the reader and to fix notation we recall some basic notions from category theory.

7.1 Categories

A *category* $\mathbb{A}$ consists of a class of objects $Obj(\mathbb{A})$, and for any objects A, B, C, there are *morphism sets* $Mor_{\mathbb{A}}(A, B)$ and $Mor_{\mathbb{A}}(B, C)$ with associative *composition*

$$Mor_{\mathbb{A}}(A, B) \times Mor_{\mathbb{A}}(B, C) \to Mor_{\mathbb{A}}(A, C), \;\; (f, g) \mapsto gf \; .$$

Sometimes the composition gf is also denoted by $g{\cdot}f$ or $g \circ f$.

A set $Mor_{\mathbb{A}}(A, B)$ may be empty, except for $A = B$, since $Mor_{\mathbb{A}}(B, B)$ always should contain an *identity morphism* 1_B satisfying $g1_B = g$ and $1_B f = f$, for $g \in Mor_{\mathbb{A}}(B, C)$, $f \in Mor_{\mathbb{A}}(A, B)$.

It is customary to write $A \in \mathbb{A}$ instead of $A \in Obj(\mathbb{A})$, and $Mor(B, C)$ instead of $Mor_{\mathbb{A}}(B, C)$ if no uncertainty can arise.

The connection between two categories is given by

7.2 Functors

A *covariant functor* $F\colon \mathbb{A} \to \mathbb{B}$ between two categories consists of assignments of $Obj(\mathbb{A}) \to Obj(\mathbb{B})$, $A \mapsto F(A)$, and of morphisms $Mor(A, B) \to Mor(A, B)$, $f \mapsto F(f)$, such that

$$F(1_A) = 1_{F(A)} \;\text{ and }\; F(fg) = F(f)F(g).$$

By definition, one has a map of sets,

$$\Phi^F_{A,A'}\colon Mor_{\mathbb{A}}(A, A') \to Mor_{\mathbb{B}}(F(A), F(A')) \,,$$

and the functor F is called *faithful* if $\Phi^F_{A,A'}$ is injective, *full* if $\Phi^F_{A,A'}$ is surjective, and *fully faithful* if $\Phi^F_{A,A'}$ is bijective.

Contravariant functors F reverse the composition of morphisms, that is, $F(fg) = F(g)F(f)$

The relation between two functors is described by

7.3 Natural transformations

Let $F, F'\colon \mathbb{A} \to \mathbb{B}$ be covariant functors. A *natural transformation* $\alpha\colon F \to F'$ is given by a family of morphisms

$$\alpha_A: F(A) \to F'(A) \text{ in } \mathbb{B},\ A \in Obj(\mathbb{A}),$$

such that any $f: A \to B$ in $\mathbb{A}$ induces commutativity of the diagram in $\mathbb{B}$,

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ F'(A) \arrow[r, "F'(f)"] & F'(B)\,. \end{tikzcd}$$

7.4 Adjoint pairs of functors

A pair (L, R) of functors $L: \mathbb{A} \to \mathbb{B}$ and $R: \mathbb{B} \to \mathbb{A}$ between categories $\mathbb{A}$ and $\mathbb{B}$ is called *adjoint* if there are bijections, natural in $A \in Obj(\mathbb{A})$ and $B \in Obj(\mathbb{B})$,

$$\vartheta_{A,B}: Mor_{\mathbb{B}}(L(A), B) \to Mor_{\mathbb{A}}(A, R(B))\ .$$

Such a pair can be characterised by the associated natural transformations

$$\textit{unit}\ \ \eta: 1_{\mathbb{A}} \to RL \quad \text{and} \quad \textit{counit}\ \ \varepsilon: LR \to 1_{\mathbb{B}}.$$

with the *trianglular identities*

$$R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R = 1_R, \quad L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L = 1_L.$$

Vector spaces with products, that is, algebras, lead to the consideration of functors with products. The products here are given by natural transformations and the rules for them are taken from the relevant properties for algebras.

7.5 Monads and their modules

A *monad* on $\mathbb{A}$ is a triple $\mathbb{F} = (F, m, \iota)$, where $F: \mathbb{A} \to \mathbb{A}$ is a functor and

$$m: FF \to F, \quad \iota: 1_{\mathbb{A}} \to F,$$

are natural transformations with commutative diagrams

$$\begin{tikzcd} FFF \arrow[r, "m_F"] \arrow[d, "Fm"'] & FF \arrow[d, "m"] \\ FF \arrow[r, "m"'] & F\,, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\iota F"] \arrow[d, "F\iota"'] \arrow[dr, "="] & FF \arrow[d, "m"] \\ FF \arrow[r, "m"'] & F\,. \end{tikzcd}$$

An $\mathbb{F}$-*module* is an object A in $\mathbb{A}$ with a morphism $\varrho_A\colon F(A) \to A$ inducing commutative diagrams

$$\begin{tikzcd} FF(A) \arrow[r, "m_A"] \arrow[d, "F\varrho_A"'] & F(A) \arrow[d, "\varrho_A"] \\ F(A) \arrow[r, "\varrho_A"'] & A\,, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "1_A"'] & F(A) \arrow[d, "\varrho_A"] \\ & A\,. \end{tikzcd}$$

Morphisms between two F-modules (A, ϱ) and (A', ϱ') are given by a morphism $h\colon A \to A'$ in $\mathbb{A}$ implying commutativity of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(h)"] \arrow[d, "\varrho"'] & F(A') \arrow[d, "\varrho'"] \\ A \arrow[r, "h"] & A'. \end{tikzcd}$$

With this morphisms, the F-modules form a category, denoted by $\mathbb{A}_F$.

Clearly, for any object A, $F(A)$ has a an F-module structure given by $m_A\colon FF(A) \to F(A)$ and for any morphism h in $\mathbb{A}$, $F(h)$ is an F-module morphism. This leads to the *free functor* and the *forgetful functor*,

$$\phi_F\colon \mathbb{A} \to \mathbb{A}_{\mathbb{F}},\ A \ \mapsto\ (F(A), m_A)\,, \quad U_F\colon \mathbb{A}_{\mathbb{F}} \to \mathbb{A},\ (A, \varrho) \mapsto A\,,$$

and (ϕ_F, U_F) form an adjoint pair by the bijections for $A \in Obj(\mathbb{A})$ and $B \in Obj(\mathbb{A}_{\mathbb{F}})$,

$$Mor_{\mathbb{A}_F}(F(A), B) \to Mor_{\mathbb{A}}(A, U_F(B))\,, \quad f \mapsto f \circ \iota_A\,.$$

Notice that $U_F\phi_F = F$.

7.6 Comonads and their comodules

A *comonad* is a triple $\mathbb{G} = (G, \delta, \varepsilon)$, where $G\colon \mathbb{A} \to \mathbb{A}$ is a functor with *coproduct* and *counit*, that is, natural transformations

$$\delta\colon G \to GG, \quad \varepsilon\colon G \to 1_{\mathbb{A}},$$

with commuting diagrams

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & GG \arrow[d, "G\delta"] \\ GG \arrow[r, "\delta_G"] & GGG\,, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, "="] & GG \arrow[d, "\varepsilon_G"] \\ GG \arrow[r, "G\varepsilon"'] & G. \end{tikzcd}$$

A $\mathbb{G}$-*comodule* is an object $A \in Obj(\mathbb{A})$ with a morphism $\rho^A\colon A \to G(A)$ in $\mathbb{A}$, inducing commutativity of the diagrams

$$\begin{tikzcd} A \arrow[r, "\rho^A"] \arrow[d, "\rho^A"'] & G(A) \arrow[d, "\delta_A"] \\ G(A) \arrow[r, "G\rho^A"] & GG(A)\,, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\rho^A"] \arrow[dr, "1_A"'] & G(A) \arrow[d, "\varepsilon_A"] \\ & A. \end{tikzcd}$$

Morphisms between comodules (A, ρ), (A', ρ') are morphisms $h\colon A \to A'$ in $\mathbb{A}$ with commutative diagrams

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\rho"'] & A' \arrow[d, "\rho'"] \\ G(A) \arrow[r, "G(h)"] & G(A'). \end{tikzcd}$$

The resulting category of G-comodules is denoted by $\mathbb{A}^G$.

Similar to the module case, for any $A \in \mathbb{A}$, $(G(A), \delta_A)$ is a G-comodule and one has the *(cofree) functor* and the *forgetful functor*,

$$\phi^G\colon \mathbb{A} \to \mathbb{A}^G,\ A \mapsto (G(A),\ \delta_A)\,, \quad U^G\colon \mathbb{A}^G \to \mathbb{A},\ (A, \rho) \mapsto A\,,$$

and (U^G, ϕ^G) form an adjoint pair of functors by the bijections

$$Mor_{\mathbb{A}^G}(B, G(A)) \to Mor_{\mathbb{A}}(U^G(B), A)\,, \quad f \mapsto \varepsilon_A \circ f\,,$$

for any $A \in Obj(\mathbb{A})$ and $B \in Obj(\mathbb{A}^G)$. Notice that $U^G\phi^G = G$.

Composing two monads one expects to get – under certain conditions – again a monad. New structures arise when a monad is combined with a comonad. For this one considers

7.7 Mixed distributive laws

Let (F, m, ι) be a monad and (G, δ, ε) a comonad on the category $\mathbb{A}$. Then a natural transformation

$$\lambda\colon FG \to GF$$

is said to be a *mixed distributive law* or *entwining* provided it induces commutativity of the diagrams

$$\begin{tikzcd} FFG \arrow[rr, "m_G"] \arrow[d, "F\lambda"'] & & FG \arrow[d, "\lambda"] \\ FGF \arrow[r, "\lambda_F"] & GFF \arrow[r, "Gm"] & GF, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\iota_G"] \arrow[dr, "G\iota"'] & FG \arrow[d, "\lambda"] \\ & GF, \end{tikzcd}$$

$$\begin{tikzcd} FG \arrow[r, "F\delta"] \arrow[d, "\lambda"'] & FGG \arrow[r, "\lambda_G"] & GFG \arrow[d, "G\lambda"] \\ GF \arrow[rr, "\delta_F"] & & GGF, \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "F\varepsilon"] \arrow[d, "\lambda"'] & F \\ GF \arrow[ur, "\varepsilon_F"'] & \end{tikzcd} .$$

In this setting, it is of interest to consider *mixed bimodules*, that is, objects $A \in \mathbb{A}$ with an F-module structure $\varrho\colon F(A) \to A$ and a G-comodule structure $\omega\colon A \to G(A)$ with commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "\varrho"] \arrow[d, "F(\omega)"'] & A \arrow[r, "\omega"] & G(A) \\ FG(A) \arrow[rr, "\lambda_A"] & & GF(A). \arrow[u, "G(\varrho)"'] \end{tikzcd}$$

Morphisms between two mixed bimodules are required to be F-module and G-comodule morphisms.

Formally one can also consider entwinings $GF \to FG$. These, however, do not relate in the same way to mixed bimodules (involves Kleisli categories, e.g. [12]).

Of particular interest are endofunctors which are both a monad as well as a comonad. In this case the question for the compatibility of the two structure arises.

7.8 Bimonads

Let $B\colon \mathbb{A} \to \mathbb{A}$ be a functor allowing for a monad (B, m, ι), a comonad (B, δ, ε), and a mixed distributive law $\lambda\colon BB \to BB$. These data are called a *bimonad* provided they induce commutativity of the diagram

$$\begin{tikzcd} BB \arrow[r, "m"] \arrow[d, "B\delta"'] & B \arrow[r, "\delta"] & BB \\ BBB \arrow[rr, "\lambda_B"] & & BBB. \arrow[u, "Bm"'] \end{tikzcd} \tag{1}$$

It is customary to call the mixed (B, B)-bimodules *Hopf modules* and we denote them by $\mathbb{A}_B^B$. Commutativity of (1) implies that, for every $A \in \mathbb{A}$, $B(A)$ is a Hopf module with

$$\text{coaction } \delta_A\colon B(A) \to BB(A) \text{ and action } m_A\colon BB(A) \to B(A)\ .$$

Thus one obtains a functor

$$\phi_B^B\colon \mathbb{A} \to \mathbb{A}_B^B\,, \quad A \mapsto (B(A), m_A, \delta_A)\ .$$

A natural transformation $S\colon B \to B$ is called an *antipode* if

$$m \cdot SB \cdot \delta = 1_B \cdot \varepsilon = m \cdot BS \cdot \delta\ ,$$

and the bimonad (B, m, δ, λ) is called a *Hopf monad* provided such an antipode exists. Under mild restrictions, this is the case if and only if ϕ_B^B is an equivalence of categories (e.g. [92], [50, 5.6]).

Over a commutative ring R, an R-bialgebra H is a Hopf algebra provided the bimonad $H \otimes_R -$ is a Hopf monad. If the Picard group of the ring R is zero, then any Hopf algebra H, with H finitely generated and projective as an R-module, has also the structure of a Frobenius algebra (see [66]).

Remark. The notion of distributive laws (of mixed type) goes back to Beck [9] and we refer to [12, 15, 92] and the references given there for further information. In more general situations, (simple) distributive laws are used by B. Klin in [42] in his study of operational semantics. He shows how *stream systems* and *Mealy machines* can be described by comodules and distributive laws.

For any monoid S, the monoid algebra $K[S]$ over a field K is a bialgebra and provides the setting for the *Myhill-Nerode Theorem* in the theory of formal languages and (finite) automata (e.g., [64, 77]).

Monads and comonads are closely related to adjoint pairs of functors (e.g. [27]):

7.9 Adjoint pairs and (co)monads

Let $L\colon \mathbb{A} \to \mathbb{B}$ and $R\colon \mathbb{B} \to \mathbb{A}$ be an adjoint pair of functors (see 7.4) with

$$\textit{unit}\ \eta\colon 1_{\mathbb{A}} \to RL \ \text{ and } \ \textit{counit}\ \varepsilon\colon LR \to 1_{\mathbb{B}}\ ,$$

Then $RL\colon \mathbb{A} \to \mathbb{A}$ has a monad structure with

$$\textit{product}\ \mu = R\varepsilon_L\colon RLRL \to RL \ \text{ and } \ \textit{unit}\ \eta\colon 1_{\mathbb{A}} \to RL\ ,$$

and $LR\colon \mathbb{B} \to \mathbb{B}$ has a comonad structure with

$$\textit{coproduct}\ \delta = L\eta_R\colon LR \to LRLR \ \text{ and } \ \textit{counit}\ \varepsilon\colon LR \to 1_{\mathbb{B}}\ .$$

Our view on monads and comonads may suggest to consider *comonads* as *dual* to *monads* and vice versa. As a special case one has the dual A^* of a finite dimensional algebra A. However, this only works because of the finite dimension since then the endofunctors $A^* \otimes_K -$ and $Hom_K(A, -)$ are isomorphic. Without finite dimension we nevertheless get that the functor $Hom_K(A, -)$ is a comonad. This can be formulated in full generality.

7.10 Adjoint endofunctors

Let (F, G) be an adjoint pair of endofunctors on a category $\mathbb{A}$ with bijection

$$\varphi_{X,Y}\colon Mor_{\mathbb{A}}(F(X), Y) \to Mor_{\mathbb{A}}(X, G(Y))\ ,$$

and $\eta\colon 1_{\mathbb{A}} \to GF$, $\varepsilon\colon FG \to 1_{\mathbb{A}}$ as unit and counit.

Assume (F, m, ι) to be a monad. Then, for $X, Y \in \mathbb{A}$, there are diagrams

$$\begin{tikzcd} Mor_{\mathbb{A}}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "Mor(m_X, Y)"'] & Mor_{\mathbb{A}}(X, G(Y)) \arrow[d, dotted, "Mor(X,?)"] \\ Mor_{\mathbb{A}}(FF(X), Y) \arrow[r, "\simeq"] & Mor_{\mathbb{A}}(X, GG(Y)), \end{tikzcd}$$

$$\begin{tikzcd} Mor_{\mathbb{A}}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "Mor(\iota_X, Y)"'] & Mor_{\mathbb{A}}(X, G(Y)) \arrow[dl, dotted, "Mor(X,??)"] \\ Mor_{\mathbb{A}}(X, Y) & \end{tikzcd},$$

in which the dotted morphisms exist by composition of the other morphisms (φ is invertible). By the *Yoneda Lemma* it follows that they are induced by morphisms

$$\underline{\delta}_Y : G(Y) \to GG(Y) \quad \text{and} \quad \underline{\varepsilon}_Y : G(Y) \to Y,$$

and these are explicitly given by the natural transformations

$$\underline{\delta} : G \xrightarrow{\eta G} GFG \xrightarrow{G\eta FG} GGFFG \xrightarrow{GGmG} GGFG \xrightarrow{GG\varepsilon} GG\,,$$
$$\underline{\varepsilon} : G \xrightarrow{eG} FG \xrightarrow{\varepsilon} 1_{\mathbb{A}}\,,$$

yielding a comonad $(G, \underline{\delta}, \underline{\varepsilon})$.

Based on this kind of arguments one obtains (e.g. [12, 2.8]):

(1) *F has a monad structure if and only if G allows for a comonad structure. In this case, the category of F-modules is isomorphic to the category of G-comodules by the functors*

$$Q : \mathbb{A}_F \to \mathbb{A}^G, \quad F(A) \xrightarrow{h} A \;\mapsto\; A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A)\,,$$
$$Q^{-1} : \mathbb{A}^G \to \mathbb{A}_F, \quad A \xrightarrow{\rho} G(A) \;\mapsto\; F(A) \xrightarrow{F(\rho)} FG(A) \xrightarrow{\varepsilon_A} A\,.$$

(2) *F has a comonad structure if and only if G allows for a monad structure. In this case the corresponding Kleisli categories are isomorphic.*

More about these structures may be found, for example, in [12, 92]. For an explicit outline for the Hom-tensor functors in module categories in this context we refer to [94].

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Annihilators of power values of generalized skew derivations on Lie ideals

Abstract: Let R be a prime ring and L be a noncommutative Lie ideal of R. Let α be an automorphism of R and f be a generalized α-derivation of R. If a is a fixed element of R such that $af(x)^n = 0$ for all $x \in L$, where n is a fixed positive integer, then $af(x) = 0$ for all $x \in R$, unless $dim_C RC = 4$.

Keywords: Prime ring; Lie ideal; generalized skew derivation; generalized derivation; automorphism; right Martindale quotient ring; two-sided Martindale quotient ring; generalized polynomial identity.

1 Introduction, Notation, and Statements of the Results

Throughout this paper unless specially stated, R always denotes a prime ring with center $Z(R)$, extended centroid C, right Martindale quotient ring Q_r, and two-sided Martindale quotient ring Q. For any $x, y \in R$, we set $[x, y] = xy - yx$.

By a derivation of R we mean an additive map d from R into itself satisfying the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let α be an automorphism of R. An α-derivation of R is an additive mapping d satisfying $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$. α-derivations are also called *skew derivations*. When $\alpha = 1$, the identity map of R, α -derivations are merely ordinary derivations. If $\alpha \neq 1$, then $1 - \alpha$ is an α–derivation. An additive mapping $f\colon R \to R$ is called generalized α-derivation if there exists an α-derivation $d\colon R \to R$ such that $f(xy) = f(x)y + \alpha(x)d(y)$ for all $x, y \in R$. If $a, b \in R$ and $\alpha \neq 1$ is an automorphism of R, then $f(x) = ax - \alpha(x)b$ is a generalized α–derivation. Moreover, if d is an α-derivation of R, then $f(x) = ax + d(x)$ is a generalized α-derivation.

In [11], A. Giambruno and I. N. Herstein proved that if R is a semiprime ring and d is a derivation of R such that $d(x)^n = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then $d = 0$. A variety of results generalizing Herstein's theorem have been obtained by a number of authors. M. Brešar [2] proved that if R is a prime ring with a nonzero derivation d and a is an element of R such that $ad(x)^n = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then $a = 0$ provided $charR \neq (n-1)!$. Later in [17], Lee and Lin

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obtained the same conclusion assuming that $ad(x)^n = 0$ for all x in some noncentral Lie ideal of R without the assumption on characteristic.

In [6] J. C. Chang proved that if R is a prime ring and f is a right generalized skew derivation of R such that $f(x)^n = 0$ for all $x \in L$ where L is a noncommutative Lie ideal of R and n is a fixed positive integer, then $f = 0$. T. K. Lee and K. S. Liu extended this result to a nonzero ideal I of a prime ring R in [18]. Simply, they proved that if R is a prime ring and g is a generalized skew derivation of R such that $g(x)^n = 0$ for all $x \in I$, where I is a nonzero ideal of R and n is a fixed positive integer, then $g = 0$.

In [3], C. M. Chang and T. K. Lee proved the following: Let R be a prime ring, L be a noncommutative Lie ideal of R, d be a nonzero derivation of R and $0 \neq a \in R$. Suppose that $ad(x)^n \in Z(R)$ for all $x \in L$, where n is a fixed positive integer. Then $dim_C RC = 4$.

Later, J. C. Chang [4] showed that if R is a prime ring, α is an automorphism of R and f is a nonzero generalized α-derivation of R such that $f(x)^n \in Z(R)$ for all $x \in I$, where I is a nonzero ideal of R and n is a fixed positive integer, then R is either commutative or is an order in a 4-dimensional simple algebra.

More recently, J. C. Chang [5] proved the following: Let R be a prime ring and $a \in R$. Let α and β be automorphisms of R and f is a generalized (α, β)-derivation; that is $f(xy) = f(x)\alpha(y) + \beta(x)d(y)$, where d is an (α, β)-derivation(i.e., d is an additive map of R and $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$). If $af(x)^n = 0$ for all $x \in R$, where n is a positive integer, then $af(x) = 0$ for all $x \in R$. Moreover if $d \neq 0$ or $f \neq 0$, then $a = 0$.

In what follows unless stated otherwise, R will be a prime ring. We denote the right Martindale quotient ring of R by Q_r and the two-sided Martindale quotient ring of R by Q. Let C be the center of Q, which is called the extended centroid of R. The definitions, the axiomatic formulations and the properties of these quotient rings can be found in [1]. Note that Q and Q_r are also prime rings with identity and C is a field. It is known that automorphisms, derivations and α-derivations of R can be uniquely extended to Q_r and Q. In [4], we know that generalized α-derivations of R can be also uniquely extended to Q_r and Q. Indeed, by Lemma 2 in [4], if f is a generalized α-derivation of R, then $f(x) = f(1)x + d(x)$ for all $x \in R$, where d is an α-derivation of R.

We use frequently the theory of generalized polynomial identities and differential identities with automorphisms (see [1, 7, 8, 9, 13, 15, 19]).

An α-derivation d of R called X-inner if $d(x) = bx - \alpha(x)b$ for some $b \in Q$, and d is called X-outer if it is not X-inner. An automorphism α is called X-inner if $\alpha(x) = qxq^{-1}$ for some invertible element $q \in Q$ and α is called X-outer if it is not X-inner.

Before presenting the results we will state the following useful remarks:

Fact 1.1 ([7], Theorem 1). *Let R be a prime ring. If δ is a nonzero skew derivation of R and $\varphi(x_1, \dots, x_n, \delta(x_1), \dots, \delta(x_n))$ is a skew differential identitiy for R then, either δ is an inner skew derivation or R satisfies the generalized polynomial identity $\varphi(x_1, \dots, x_n, y_1, \dots, y_n)$.*

Fact 1.2 ([21], Remark 2.1). *Let R be a prime ring, U be the Utumi quotient ring of R, I be a nonzero ideal of R and $a, b \in U/\{0\}$. Let n be a fixed positive integer and δ be a*

nonzero generalized derivation of R. Suppose that $a\big(\delta(x)b\big)^n = 0$ for all $x \in I$. Then there exist $a_1, b_1 \in U$ such that $\delta(x) = a_1x + xb_1$ for all $x \in R$ and $b_1b = 0$. Moreover, either $ba_1 = 0$ or $aa_1 = 0$.

We are now ready to state the main result:

Theorem 1.1. *Let R be a prime ring, Q be its Martindale quotient ring, C be its extended centroid and α is an automorphism of R. Let f be a generalized α-derivation of R, L be a noncommutative Lie ideal of R. Suppose that $af(x)^n = 0$ for all $x \in L$, where n is a fixed positive integer, then $af(x) = 0$ for all $x \in R$, unless $dim_C RC = 4$.*

2 The results

We need the following lemmas in the sequel. In the light of Corollary in [16], we give the ensuant lemma as a consequence:

Lemma 2.1. *Let R be a noncommutative prime ring, n is a fixed positive integer and $a, b \in R$.*
(i) If $a\big([x, y]b\big)^n = 0$ for all $x, y \in R$, then $a = 0$ or $b = 0$.
(ii) If $a\big(b[x, y]\big)^n = 0$ for all $x, y \in R$, then $ab = 0$.

Lemma 2.2. *Let R be a prime ring and $a, b, c, q \in R$ with q invertible in R such that*

$$a(b[x, y] - q[x, y]q^{-1}c)^n = 0$$

for all $x, y \in R$. If R does not satisfy any nontrivial generalized polynomial identity, then $a = 0$ or $q^{-1}c \in C$ and $a(b - c) = 0$.

Proof. Assume that $a \neq 0$. If $q^{-1}c \in C$, then $a((b - c)[x, y])^n = 0$ for all $x, y \in R$. This implies that $a(b - c) = 0$ by Lemma 2.1. So we may assume that $q^{-1}c \notin C$.

Let $T = Q *_C C\{X, Y\}$ be the free product over C of the C-algebra Q and the free C-algebra $C\{X, Y\}$, with the non-commuting indeterminates X, Y. Since R does not satisfy any nontrivial generalized polynomial identity (GPI for short), we have that

$$G(X, Y) = a(b[X, Y] - q[X, Y]q^{-1}c)^n = 0 \in T.$$

Suppose for the moment that aq and $q^{-1}cq$ are C– independent. Then $a(q[X, Y]q^{-1}c)^n$ occurs nontrivially in $G(X, Y)$. If aq and $q^{-1}cq$ are linearly C–dependent, then there exits some $\lambda \in C$ such that $q^{-1}cq = \lambda aq$ and so treating the same monomial $a(q[X, Y]q^{-1}c)^n$, we obtain the monomial $a(q[X, Y]\lambda a)^n$. Assume that $\lambda a(q[X, Y]a)^n = 0 \in T$. Since $aq \neq 0$ then R is commutative, a contradiction. So $a(q[X, Y]q^{-1}c)^n$ occurs nontrivially in $G(X, Y)$. □

Lemma 2.3. *Let R be a prime ring and L be a noncommutative Lie ideal of R. Let $a, b, c, q \in R$ with q invertible in R. Suppose that $a(bx - qxq^{-1}c)^n = 0$ for all $x \in L$, where n is a fixed positive integer. Then $a = 0$ or $q^{-1}c \in C$ and $a(b - c) = 0$, unless $dim_C RC = 4$.*

Proof. If R is a domain or $a = 0$, then there is nothing to prove. So we may assume that R is not domain and $a \neq 0$. Set $I = R[L, L]R$, then in view of [12] $0 \neq [I, R] \subseteq L$. By the assumption, we have

$$a(bx - qxq^{-1}c)^n = 0 \tag{1}$$

for all $x \in [I, R]$. Since I, R and Q satisfy the same generalized polynomial identities over Q, (1) still holds for all $x \in [Q, Q]$ by [7]. Hence we may assume that $R = I = Q$ and R is centrally closed prime ring. If R is not a GPI-ring, then by Lemma 2.2 we are done. So we may assume that R is a GPI-ring. Thus R is primitive ring which is isomorphic to a dense ring of linear transformations of vector space V over a division ring D. If $dim_D V = \infty$, then in view of Lemma 2 in [20], we arrive at $a(bx - qxq^{-1}c)^n = 0$ for all $x \in R$. In particular, $a(bq^{-1}x - xq^{-1}c)^n = 0$ for all $x \in R$ and so $a(bq^{-1}x - xq^{-1}c) = 0$ for all $x \in R$ by Theorem A in [5]. Thus we have $abq^{-1}x - axq^{-1}c = 0$ for all $x \in R$. It follows from [19] that $q^{-1}c \in C$. Hence $a((b - c)x)^n = 0$ for all $x \in R$ and $a(b - c) = 0$ by Theorem 2 in [10].

Now consider the case $dim_D V < \infty$. Hence R is isomorphic to D_m, the matrix ring over D for some m. If C is finite, then D (being finite dimensional over C) is a finite ring and thus is a field by Wedderburn's theorem. In this case $R \cong C_m$. On the other hand if C is infinite and F is the maximal subfield of D, then by a standard argument (see, for instance proposition in [14]), $a(b[x, y] - q[x, y]q^{-1}c)^n = 0$ for all $x, y \in R \otimes_C F$. But $R \otimes_C F \cong D_m \otimes_C F \cong (D \otimes_C F)_m \cong F_k$ for some k. In either case, we may suppose that $R \cong F_k$ for some $k > 1$.

Assume $dim_C RC \neq 4$ then we may consider that $k \geq 3$. Let V be a k-dimensional F-linear transformations of V. For any given $v \in V$, we claim that v and $q^{-1}cv$ are F-dependent. Suppose, on the contrary that v and $q^{-1}cv$ are F-independent. Since $dim_F V \geq 3$, we can choose $w \in V$ such that v, $q^{-1}cv$ and w are F-independent. By the density of R, there exists $x, y \in R$ such that

$$xq^{-1}cv = 0, xv = 0$$

$$yq^{-1}cv = w, yv = 0, xw = q^{-1}v\ .$$

This implies that $[x, y]v = 0$ and $[x, y]q^{-1}cv = q^{-1}v$. Thus

$$0 = a(b[x, y] - q[x, y]q^{-1}c)^n v = (-1)^n av$$

and so $av = 0$. Hence we prove that if $av \neq 0$, then v and $q^{-1}cv$ are C-dependent. Since $a \neq 0$, there exists $v_0 \in V$ such that $av_0 \neq 0$. Thus $q^{-1}cv_0$ and v_0 are F-dependent and $q^{-1}cv_0 = \lambda v_0$ for some $\lambda \in F$. We claim that v and $q^{-1}cv$ are F-dependent for all $v \in V$. For $v \in V$, by the previous proof, we may assume that v and v_0 are linearly

independent over F and $av = 0$. Since $dim_F V \geq 3$, we can choose v, v_0 and w are linearly independent over F. Suppose first that $aw \neq 0$. Then $a(v + v_0) = av_0 \neq 0$ and $a(v + w) = aw \neq 0$. So there exist β, γ and $\varrho \in F$ such that $q^{-1}cw = \beta w$, $q^{-1}c(v + v_0) = \gamma(v+v_0)$, and $q^{-1}c(v+w) = \varrho(v+w)$. Comparing the equations above with $q^{-1}cv_0 = \lambda v_0$ we have $\lambda = \beta = \gamma = \varrho$. This implies that $q^{-1}cv = \lambda v$ and so v and $q^{-1}cv$ are F-dependent.

Now suppose that $aw = 0$. Replacing w by $v_0 + w$ and using the independence of $\{v, v_0, w\}$ we get a contradiction. Hence $aw \neq 0$. By the proof above, v and $q^{-1}cv$ are F-dependent. So in any case, v and $q^{-1}cv$ are F-dependent for all $v \in V$. By the standard argument, it is easy to see that $q^{-1}c \in F$. In this manner $a((b - c)[x, y])^n = 0$ for all $x, y \in R$. It follows from Lemma 2.1 that $a(b - c) = 0$. ☐

Lemma 2.4. *Let R be a prime ring and L be a noncommutative Lie ideal of R. Let $a, b, c \in R$ and α be an automorphism of R. Suppose that $a(bx - \alpha(x)c)^n = 0$ for all $x \in L$, where n is a fixed positive integer. Then either $a = 0$ or $a(bx - \alpha(x)c) = 0$ for all $x \in R$, unless $dim_C RC = 4$.*

Proof. If R is a domain or $a = 0$, then there is nothing to prove. So we may assume that R is not a domain and $a \neq 0$. Set $I = R[L, L]R$ and by [12] $0 \neq [I, R] \subseteq L$. So we have

$$a(f(x))^n = 0 \tag{2}$$

for all $x \in [I, R]$, where $f(x)$ stands for $bx - \alpha(x)c$. In the light of Theorem 1 in [9], (2) holds for all $x \in [Q, Q]$. So we may assume thar $R = I = Q$ and R is centrally closed prime ring. If α is an X-inner automorphism of R, that is there exists $0 \neq q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$, then we are done by Lemma 2.3. In the case that α is X-outer, also we have

$$a(bx - \alpha(x)c)^n = 0 \tag{3}$$

for all $x \in [R, R]$. If b or c is zero, we are done by Lemma 2.1. So we consider that b and c are nonzero. By the main theorem in [8], R is a GPI-ring. Thus R is a primitive ring with nonzero socle by Theorem 3 in [19]. Since R is not a domain, R has nontrivial idempotents. Let e be a nontrivial idempotent of R. Applying (3) we have

$$\begin{aligned} 0 &= a\big(b\big[\alpha^{-1}(1 - e)xe, \alpha^{-1}(1 - e)ye\big] \\ &\quad - \big[(1 - e)\alpha(x)\alpha(e), (1 - e)\alpha(y)\alpha(e)\big]c\big)^n(1 - e) \\ &= a\big(\big[(1 - e)\alpha(x)\alpha(e), (1 - e)\alpha(y)\alpha(e)\big]c\big)^n(1 - e) \\ &= a(1 - e)\big((\alpha(x)\alpha(e)(1 - e)\alpha(y) - \alpha(y)\alpha(e)(1 - e)\alpha(x))\alpha(e)c(1 - e)\big)^n \end{aligned}$$

for all $x, y \in R$. This implies that

$$0 = a(1 - e)\big((x\alpha(e)(1 - e)y - y\alpha(e)(1 - e)x)\alpha(e)c(1 - e)\big)^n \tag{4}$$

for all $x, y \in R$. We may assume that $\alpha(e)(1 - e) \neq 0$. By Remark 2.1 in Xu et al. [21] it is clear that

$\alpha(e)(1-e)R\alpha(e)c(1-e) \subseteq C\alpha(e)c(1-e)$ and either
$a(1-e)R\alpha(e)(1-e) \subseteq Ca(1-e)$ or
$\alpha(e)c(1-e)R\alpha(e)(1-e) \subseteq C\alpha(e)c(1-e)$. Hence by (4) we obtain

$$\begin{aligned} 0 =& a(1-e)\big(x\lambda\alpha(e)c(1-e) - y\mu\alpha(e)c(1-e)\big) \\ &\big(x\alpha(e)(1-e)y - y\alpha(e)(1-e)x)\alpha(e)c(1-e)\big)^{n-1} \end{aligned} \tag{5}$$

for some $\lambda, \mu \in C$. Continuing the same process in (5) we have

$$a(1-e)\big((\lambda x - \mu y)\alpha(e)c(1-e)\big)^n = 0$$

for all $x, y \in R$. Hence

$$a(1-e)\big(RC\alpha(e)c(1-e)\big)^n = 0$$

Since RC is a prime ring then either $a(1-e) = 0$ or $\alpha(e)c(1-e) = 0$. Now assume that e is an idempotent of R with $\alpha(e)(1-e) = 0$, then we see again from (3) that

$$\begin{aligned} 0 &= a(b[\alpha^{-1}(1-e)xe, ye] - [(1-e)\alpha(x)\alpha(e), \alpha(y)\alpha(e)]c)^n(1-e) \\ &= a(1-e)(\alpha(x)\alpha(e)\alpha(y)\alpha(e)c(1-e))^n \end{aligned}$$

for all $x, y \in R$. By Theorem 2 in [10] we arrive at $a(1-e) = 0$ or $\alpha(e)R\alpha(e)c(1-e) = 0$. By the primeness of R we get $a(1-e) = 0$ or $\alpha(e)c(1-e) = 0$. We will adopt the proof of Lemma 4 in [5] with some modification. Assume that $a(1-e) = 0$ for some nontrivial idempotent $e \in R$. Since $e + (1-e)xe$ is also an idempotent for all $x \in R$ and $a\big(e + (1-e)xe\big) = ae = a \neq 0$ then

$$\alpha\big(1 - e - (1-e)xe\big)c\big(e + (1-e)xe\big) = 0 \tag{6}$$

for all $x \in R$. Besides, if $\alpha(e)c(1-e) = 0$ for all idempotents in R then (6) holds for all idempotents in R. Expanding (6) we obtain

$$\alpha(1-e)ce - \alpha(1-e)\alpha(x)\alpha(e)ce + \alpha(1-e)c(1-e)xe - \alpha(1-e)\alpha(x)\alpha(e)c(1-e)xe = 0 \tag{7}$$

for all $x, y \in R$. In particular, we have $\alpha(e)ce = ce$. Hence (7) reduces to

$$(1-\alpha(e))c(1-e)xe - (1-\alpha(e))\alpha(x)ce - (1-\alpha(e))\alpha(x)\alpha(e)c(1-e)xe = 0 \tag{8}$$

for all $x \in R$. Linearizing (8) we obtain

$$(1-\alpha(e))\big(\alpha(x)\alpha(e)c(1-e)y + \alpha(y)\alpha(e)c(1-e)x\big)e = 0\,.$$

Since α is an X-outer automorphism of R, then

$$(1-\alpha(e))x\alpha(e)c(1-e)xe = 0$$

by [13]. Using the primeness of R we get $\alpha(e)c(1-e) = 0$ and thus (8) implies

$$c(1-e)xe - (1-\alpha(e))\alpha(x)ce = 0$$

for all $x \in R$. Again, in the light of [13]

$$c(1-e)xe - (1-\alpha(e))\alpha(x)ce = 0$$

for all $x, y \in R$. Hence $c(1-e) = 0$ which implies $c = 0$. Eventually $a(bx)^n = 0$ for all $x \in [R, R]$ and the proof is completed by Lemma2.1. ☐

Proof of the Main Theorem. If $a = 0$, then there is nothing to prove. So we may assume that $a \neq 0$. Set $I = R[L, L]R$. Then $0 \neq [I, R] \subseteq L$ by [12]. It is known that there exists $s = f(1) \in Q_r$ such that $f(x) = sx + \delta(x)$ for all $x \in R$, where δ is an α-derivation of R by Lemma 2 in [4]. By the assumption, we have

$$a(sx + \delta(x))^n = 0 \tag{9}$$

for all $x \in L$ and hence for all $x \in [I, R]$. By Theorem 2 in [7], I, R and Q_r satisfy the same GPIs with single skew derivation over Q_r and hence (9) holds for all $x \in [Q_r, Q_r]$. So we may assume that $R = I = Q_r$. If δ is X-inner, then $\delta(x) = bx - \alpha(x)b$ for all $x \in R$, where $b \in Q$ and α is an automorphism of R. We rewrite (9) as

$$a((s+b)x - \alpha(x)b)^n = 0$$

for all $x \in [R, R]$. So we are done by Lemma 2.4. Finally we consider the case that δ is X-outer. Using (9) we arrive at

$$\begin{aligned} 0 &= a(s[x,y] + \delta[x,y])^n \\ &= a(s[x,y] + \delta(xy) - \delta(yx))^n \\ &= a(s[x,y] + \delta(x)y + \alpha(x)\delta(y) - \delta(y)x - \alpha(y)\delta(x))^n \end{aligned}$$

for all $x, y \in R$. By Fact 1 we have

$$a(s[x,y] + zy + \alpha(x)u - ux - \alpha(y)z)^n = 0 \tag{10}$$

for all $x, y, z, u \in R$. Setting $x = 0$ in (10), we see that $a(zy - \alpha(y)z)^n = 0$ for all $y, z \in R$. In view of Lemma 4 in [4] we have $a(zy - \alpha(y)z) = 0$ for all $y, z \in R$. Right multiplying by r in the last equation yields that $azyr - \alpha(y)zr = 0$ for all $r, y, z \in R$. On the other hand, $azyr - \alpha(yr)z = 0$ for all $r, y, z \in Q$. Comparing the last two relations, it follows that $a\alpha(y)(\alpha(r)z - zr) = 0$ for all $r, z, y \in R$. Since $a \neq 0$ and R is a prime ring, then $\alpha(r)z = zr$ for all $r, z \in R$. In particular $\alpha(r) = r$ for all $r \in R$ which implies $rz = zr$ for all $r, z \in R$. So R is commutative, a contradiction.
☐

Corollary 2.1. *Let R be a prime ring and L be a noncommutative Lie ideal of R. Suppose that $a \in R$ and f is a nonzero generalized α-derivation of R such that $af(x)^n = 0$ for all $x \in L$, where $n \geq 1$ a fixed positive integer. Then $a = 0$ or $f(x) = qx$ for all $x \in R$, where $q \in Q$ and $aq = 0$, unless $dim_C RC = 4$.*

Proof. Suppose that $af(x)^n = 0$ for all $x \in L$, then $af(x) = 0$ for all $x \in R$, unless $dim_C RC = 4$ by Theorem. Thus,

$$0 = af(xy) = a\big(f(x)y + \alpha(x)\delta(y)\big) = a\alpha(x)\delta(y)$$

for all $x, y \in R$, where δ is an α-derivation of R. Using the primeness of R, we have either $a = 0$ or $\delta = 0$. If $\delta = 0$, then $f(x) = qx$ for some $q \in Q_r$ by Lemma 2 in [4] and hence we get $a(qx) = 0$ for all $x \in R$. This implies that $aq = 0$ and the proof is finished. □

Since every α-derivation d of R is a generalized α-derivation of R, we conclude with:

Corollary 2.2. *Let R be a prime ring and L be a noncommutative Lie ideal of R. Suppose that $a \in R$ and d is an α-derivation such that $ad(x)^n = 0$ for all $x \in L$. Then $a = 0$ or $d = 0$, unless $dim_C RC = 4$.*

Example 1. *Let F be a field and $q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be an invertible element of the ring $R = \left\{ \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} | u, v, w \in F \right\}$. Suppose that*

$$\alpha(x) = qxq^{-1} = \begin{pmatrix} u & -u + v + w \\ 0 & w \end{pmatrix}$$

for all $x \in R$.

$$G(x) = cx - \alpha(x)d = \begin{pmatrix} 0 & u - v \\ 0 & 0 \end{pmatrix}$$

is a nonzero generalized α-derivation of R where $c = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R$. If $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$ then $aG(x)^n = 0$ for all $x \in R$ where n is a fixed positive integer. But it easy to see that

$$aG(x) == \begin{pmatrix} 0 & u - v \\ 0 & 0 \end{pmatrix} \neq 0$$

unless $u = v$.

Example 2. *Let F be a field with $char\ F = 2$ and*

$$R = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$$

be a ring. $L = \left\{ \begin{pmatrix} \beta & \lambda \\ \lambda & \beta \end{pmatrix} : \lambda, \beta \in F \right\}$ is a Lie ideal of R and let $q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$ be an invertible element. Suppose that

$$\alpha(x) = qxq^{-1} = \begin{pmatrix} u + z & u + v + z + w \\ z & z + w \end{pmatrix}$$

for all $x = \begin{pmatrix} u & v \\ z & w \end{pmatrix} \in R$.

$$G(x) = cx - \alpha(x)d = \begin{pmatrix} 0 & u+v+z \\ u+z & v+z+w \end{pmatrix}$$

is a nonzero generalized α*-derivation of* R *where* $c = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$. *If* $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *then* $aG(x)^n = 0$ *for all* $x \in L$ *where* n *is a fixed positive integer, but*

$$aG(x) = \begin{pmatrix} u+z & v+z \\ 0 & 0 \end{pmatrix} \neq 0 .$$

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Omar Ait Zemzami, Abdellah Mamouni, and Lahcen Oukhtite

Generalized derivations on prime rings with involution

Abstract: In the present paper, we study some commutativity criteria for a prime ring with involution (R, $*$) which admits generalized derivations F and G satisfying various identities. Further, examples are given to demonstrate that the restrictions imposed on the hypothesis of our results are not superfluous.

Keywords: Prime ring; involution; commutativity; derivation; generalized derivation.

1 Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive map $* : R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denote by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq (0)$. An element x is *normal* if $xx^* = x^*x$. If all elements in R are normal, then R is called a *normal ring* (or equivalently, $*$ is commuting).

An additive mapping $d: R \longrightarrow R$ is a *derivation* on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$ be a fixed element. A map $d: R \longrightarrow R$ defined by $d(x) = [a, x] = ax - xa$, $x \in R$, is a derivation on R, which is called an *inner derivation* defined by a. In [5], Brešar introduced the notion of generalized derivations in rings: an additive mapping $F: R \longrightarrow R$ is called generalized derivation if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Obviously, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier (i.e., an additive mapping satisfying $f(xy) = f(x)y$ for all $x, y \in R$). Basic examples of generalized derivations are the following: (i) $F(x) = ax + xb$ for $a, b \in R$; (ii) $F(x) = ax$ for some $a \in R$.

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Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. Recently many authors have studied commutativity of prime and semi-prime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the rings (see [1]-[4],[6],[10] and [11] for further details).

Motivated by some results in the literature, especially recent work of S. Ali and al. (see [2] and [3] for further references), here we continue the line of investigation regarding the study of commutativity for a ring with involution of the second kind provided with generalized derivations satisfying various identities.

2 Main results

The following Lemmas are essential for developing the proofs of our results.

Lemma 2.1 ([8], Fact 1). *Let $(R, *)$ be a 2-torsion free prime ring with involution provided with a derivation d. Then $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ implies that $d(z) = 0$ for all $z \in Z(R)$.*

Lemma 2.2. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If there exists a derivation d of R such that one of the following hold:*
(1) $(xy + y^*x^*)d(h)^2 \in Z(R)$ for all $x, y \in R$ and $h \in Z(R) \cap H(R)$
(2) $(xy + y^*x^*)d(h) \in Z(R)$ for all $x, y \in R$ and $h \in Z(R) \cap H(R)$
then R is commutative or $d(Z(R)) = \{0\}$.

Proof. Assume that $(xy + y^*x^*)d(h)^2 \in Z(R)$ for all $x, y \in R$ and $h \in Z(R) \cap H(R)$. Using the fact that R is prime, it follows that either $xy + y^*x^* \in Z(R)$ for all $x, y \in R$ or $d(h)^2 = 0$ for all $h \in Z(R) \cap H(R)$.
Suppose $xy + y^*x^* \in Z(R)$ for all $x, y \in R$; in particular taking $y = s$ where $s \in Z(R) \cap S(R)\backslash\{0\}$, we get $(x - x^*)s \in Z(R)$. Therefore $x - x^* \in Z(R)$ for all $x \in R$ and thus $[x, x^*] = 0$ for all $x \in R$. This further implies that R is normal and ([3], Lemma 1) assures that R is commutative.
If $d(h)^2 = 0$ for all $h \in Z(R) \cap H(R)$, then $d(h) = 0$ for all $h \in Z(R) \cap H(R)$ in which case Lemma 2.1 yields that $d(z) = 0$ for all $z \in Z(R)$.
Now suppose $(xy + y^*x^*)d(h) \in Z(R)$ for all $x, y \in R$ and $h \in Z(R) \cap H(R)$; then using the primeness of R, we get either $xy + y^*x^* \in Z(R)$ or $d(h) = 0$.
If $xy + y^*x^* \in Z(R)$ for all $x, y \in R$, then using a similar proof as above, we may conclude that R is commutative.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then by virtue of Lemma 2.1 we are forced to conclude that $d(z) = 0$ for all $z \in Z(R)$. □

Theorem 2.1. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $(F, d)$and (G, g) are nonzero generalized derivations, then the following assertions are equivalent:*
(1) $G(xx^*) + F(x)F(x^*) \pm x^*x \in Z(R)$ for all $x \in R$;
(2) $G(xx^*) - F(x)F(x^*) \pm x^*x \in Z(R)$ for all $x \in R$;
(3) $G(xx^*) + F(x^*)F(x) \pm x^*x \in Z(R)$ for all $x \in R$;
(4) $G(xx^*) - F(x^*)F(x) \pm x^*x \in Z(R)$ for all $x \in R$;
(5) R is commutative.

Proof. It is obvious that (5) implies (1), (2), (3) and (4). So we need to prove that (1) $\Longrightarrow$ (5), (2) $\Longrightarrow$ (5), (3) $\Longrightarrow$ (5), and (4) $\Longrightarrow$ (5).
(1) $\Longrightarrow$ (5) By the assumption, we have

$$G(xx^*) + F(x)F(x^*) + x^*x \in Z(R) \quad \text{for all} \quad x \in R\,. \tag{1}$$

A linearization of (1) yields that

$$G(xy^*) + G(yx^*) + F(x)F(y^*) + F(y)F(x^*) + x^*y + y^*x \in Z(R)\,. \tag{2}$$

First we assume that $g = d = 0$. Then substituting ys for y in (2), where $0 \neq s \in Z(R) \cap S(R)$, we get

$$G(xy) + F(x)F(y) + yx \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{3}$$

Using ([11], Theorem 1), the last expression forces us to conclude that R is commutative.
If $g = 0$ and $d \neq 0$, then writing $y = yh$ in (2), where $0 \neq h \in Z(R) \cap H(R)$, we get

$$(F(x)y^* + yF(x^*))d(h) \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{4}$$

Replacing x by xh, with $0 \neq h \in Z(R) \cap H(R)$, the last expression leads to

$$(xy + y^*x^*)d(h)^2 \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{5}$$

By Lemma 2.2, we conclude that R is commutative or $d(Z(R)) = \{0\}$.
In the latter case, setting $y = ys$ in (2), where $0 \neq s \in Z(R) \cap S(R)$, we arrive at

$$G(xy) + F(x)F(y) + yx \in Z(R) \quad \text{for all} \quad x, y \in R \tag{6}$$

and ([11], Theorem 1) gives that R is commutative ring.
If $g \neq 0$ and $d = 0$, then substituting yh for y in (2), where $h \in Z(R) \cap H(R) \backslash \{0\}$, we may write

$$(xy + y^*x^*)g(h) \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{7}$$

Applying Lemma 2.2 again, it follows that R is commutative or $g(Z(R)) = \{0\}$.
Suppose that $g(z) = 0$ for all $z \in Z(R)$; taking $y = ys$ in (2), where $0 \neq s \in Z(R) \cap S(R)$, we obtain

$$G(xy) + F(x)F(y) + yx \in Z(R) \quad \text{for all} \quad x, y \in R \tag{8}$$

so R is commutative by ([11], Theorem 1).
Now we consider the remaining case $g \neq 0$ and $d \neq 0$; putting $y = yh$ in (2), where $0 \neq h \in Z(R) \cap H(R)$, we find that

$$(xy^* + yx^*)g(h) + (F(x)y^* + yF(x^*))d(h) \in Z(R) \quad \text{for all } x, y \in R\,. \tag{9}$$

Replacing x by xh in (9), with $0 \neq h \in Z(R) \cap H(R)$, we arrive at

$$(xy + y^*x^*)d(h)^2 \in Z(R) \quad \text{for all } x, y \in R\,. \tag{10}$$

Invoking Lemma 2.2, either R is commutative or $d(Z(R)) = \{0\}$.
In the latter case equation (9) reduces to

$$(xy + y^*x^*)g(h) \in Z(R) \quad \text{for all } x, y \in R \tag{11}$$

and Lemma 2.2 implies that either R is commutative or $g(Z(R)) = \{0\}$.

Suppose that $g(z) = 0$ for all $z \in Z(R)$, then replacing y by ys in (2), where $0 \neq s \in Z(R) \cap S(R)$, we get

$$G(xy) + F(x)F(y) + yx \in Z(R) \quad \text{for all } x, y \in R \tag{12}$$

and we are done by ([11], Theorem 1).

Arguing as above, with slight modifications, it is obvious to show that $G(xx^*) + F(x)F(x^*) - x^*x \in Z(R)$ for all $x \in R$ implies that R is commutative.
(2) $\Longrightarrow$ (5) We are assuming that

$$G(xx^*) - F(x)F(x^*) \pm x^*x \in Z(R) \quad \text{for all } x \in R \tag{13}$$

and therefore

$$-G(xx^*) + F(x)F(x^*) \pm x^*x \in Z(R) \quad \text{for all } x \in R\,. \tag{14}$$

Let $G' = -G$, it is obvious that G' is a generalized derivation associated with the derivation $g' = -g$ such that

$$G'(xx^*) + F(x)F(x^*) \pm x^*x \in Z(R) \quad \text{for all } x \in R\,. \tag{15}$$

Since (15) is exactly the hypothesis of (1), then R is commutative.
(3) $\Longrightarrow$ (5) By the assumption, we have

$$G(xx^*) + F(x^*)F(x) + x^*x \in Z(R) \quad \text{for all } x \in R\,. \tag{16}$$

A linearization of (16) implies that

$$G(xy^*) + G(yx^*) + F(x^*)F(y) + F(y^*)F(x) + x^*y + y^*x \in Z(R) \quad \text{for all } x, y \in R\,. \tag{17}$$

Taking yh instead of y, where $0 \neq h \in Z(R) \cap H(R)$, we find that

$$(xy^* + yx^*)g(h) + (F(x^*)y + y^*F(x))d(h) \in Z(R) \quad \text{for all } x, y \in R\,. \tag{18}$$

Replacing x by xh, where $h \in Z(R) \cap H(R) \setminus \{0\}$, we get

$$(xy + y^* x^*) d(h)^2 \in Z(R) \quad \text{for all} \quad x, y \in R \,. \tag{19}$$

Using Lemma 2.2, we conclude that R is commutative or $d(Z(R)) = \{0\}$.
In the latter case equation (18) becomes

$$(xy + y^* x^*) g(h) \in Z(R) \quad \text{for all} \quad x, y \in R \tag{20}$$

and Lemma 2.2 implies that R is commutative or $g(Z(R)) = \{0\}$.
If $g(Z(R)) = \{0\}$, then replacing y by ys in (17) where $0 \neq s \in Z(R) \cap S(R)$ we obtain

$$G(xy) + F(y)F(x) + yx \in Z(R) \quad \text{for all} \quad x, y \in R \,. \tag{21}$$

Hence ([11], Theorem 3) assures that R is commutative, or $d = g = 0$.
If $d = g = 0$, then substituting xt for x in (21), where $t \in R$, one can see that

$$G(x)ty + F(y)F(x)t + yxt \in Z(R) \quad \text{for all} \quad x, y, t \in R \tag{22}$$

that is

$$G(x)ty - G(xy)t + (G(xy) + F(y)F(x) + yx)t \in Z(R) \,. \tag{23}$$

Using (21) together with (23) we conclude that

$$[G(x)[t, y], t] = 0 \quad \text{for all} \quad x, y, t \in R \tag{24}$$

thereby obtaining

$$G(x)\Big[[t, y], t\Big] + [G(x), t][t, y] = 0 \quad \text{for all} \quad x, y, t \in R \tag{25}$$

which leads to

$$G(x)\Big[[y, t], t\Big] + [G(x), t][y, t] = 0 \quad \text{for all} \quad x, y, t \in R \,. \tag{26}$$

Substituting xu for x, where $u \in R$, we find that

$$G(x)u\Big[[y, t], t\Big] + [G(x)u, t][y, t] = 0 \quad \text{for all} \quad x, y, t, u \in R \,. \tag{27}$$

Thus we have

$$G(x)u\Big[[y, t], t\Big] + G(x)[u, t][y, t] + [G(x), t]u[y, t] = 0 \,. \tag{28}$$

In particular for $u = G(x)$, according to (26), we obtain

$$[G(x), t]G(x)[y, t] = 0 \quad \text{for all} \quad x, y, t \in R \tag{29}$$

which reduces to

$$[G(x), t]G(x)R[y, t] = 0 \quad \text{for all} \quad x, y, t \in R \,. \tag{30}$$

In view of the primeness, the last equation assures that $[G(x), t]G(x) = 0$ or $[y, t] = 0$ for all $x, y, t \in R$. Hence R is the union of the subgroups:

$$H = \{t \in R|[r, t] = 0 \text{ for all } r \in R\} \text{ and } K = \{t \in R|[G(x), t]G(x) = 0 \text{ for all } x \in R\} .$$

Accordingly, either $R = H$ or $R = K$. Since the first case implies that R is commutative, then we need only consider the case $[G(x), t]G(x) = 0$ for all $x, t \in R$ and thus $[G(x), t]RG(x) = 0$ which leads to $[G(x), t] = 0$ for all $x, t \in R$. Therefore, because of $g = 0$, it follows that $G(x)R[y, t] = 0$ for all $x, y, t \in R$. Using the fact that $G \neq 0$, then we conclude that R is commutative.

Using a similar proof, with slight modifications, it is obvious to show that $G(xx^*) + F(x^*)F(x) - x^*x \in Z(R)$ for all $x \in R$ yields that R is commutative.

(4) $\Longrightarrow$ (5) We are given that

$$G(xx^*) - F(x^*)F(x) \pm x^*x \in Z(R) \quad \text{for all } x \in R \tag{31}$$

thereby

$$-G(xx^*) + F(x^*)F(x) \pm x^*x \in Z(R) \quad \text{for all } x \in R . \tag{32}$$

Consider $G' = -G$, the last expression can be written as

$$G'(xx^*) + F(x^*)F(x) \pm x^*x \in Z(R) \quad \text{for all } x \in R . \tag{33}$$

Since (33) is exactly the hypothesis of (3), then R is commutative. This completes the proof of the theorem. □

Corollary 2.1. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If (F, d) and (G, g) are nonzero generalized derivations, then the following assertions are equivalent:*
(1) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in R$;
(2) $G(xy) \pm F(y)F(x) \pm yx \in Z(R)$ for all $x, y \in R$;
(3) R is commutative.

Theorem 2.2. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If (F, d) and (G, g) are non zero generalized derivations, then the following assertions are equivalent:*
(1) $G(xx^*) + F(x^*)F(x) \in Z(R)$ for all $x \in R$;
(2) $G(xx^*) - F(x^*)F(x) \in Z(R)$ for all $x \in R$;
(3) $G(xx^*) + F(x)F(x^*) \pm [x, x^*] \in Z(R)$ for all $x \in R$
(4) $G(xx^*) - F(x)F(x^*) \pm [x, x^*] \in Z(R)$ for all $x \in R$
(5) R is commutative.

Proof. It is obvious that (5) implies (1), (2), (3) and (4). So we need to prove that (1) $\Longrightarrow$ (5), (2) $\Longrightarrow$ (5), (3) $\Longrightarrow$ (5), and (4) $\Longrightarrow$ (5).

(1) $\Longrightarrow$ (5) Assume that

$$G(xx^*) + F(x^*)F(x) \in Z(R) \quad \text{for all } x \in R . \tag{34}$$

Linearizing equation (34), one can see that

$$G(xy^*) + G(yx^*) + F(x^*)F(y) + F(y^*)F(x) \in Z(R) \quad \text{for all } x, y \in R . \tag{35}$$

If $g \neq 0$ and $d \neq 0$, then replacing y by yh, where $0 \neq h \in Z(R) \cap H(R)$, we obtain

$$(xy^* + yx^*)g(h) + (F(x^*)y + y^*F(x))d(h) \in Z(R) \quad \text{for all } x, y \in R . \tag{36}$$

Writing xh instead of x, where $0 \neq h \in Z(R) \cap H(R)$, we get

$$(xy + y^*x^*)d(h)^2 \in Z(R) \quad \text{for all } x, y \in R . \tag{37}$$

Using Lemma 2.2 together with the preceding equation, either R is commutative or $d(Z(R)) = \{0\}$.
In the second case, because of $d(h) = 0$, equation (36) can be rewritten as

$$(xy + y^*x^*)g(h) \in Z(R) \quad \text{for all } x, y \in R . \tag{38}$$

Applying Lemma 2.2 again, we obtain R is commutative or $g(z) = 0$ for all $z \in Z(R)$. In the latter case replacing y by ys in (35), where $0 \neq s \in Z(R) \cap S(R)$, we arrive at

$$G(xy) + F(y)F(x) \in Z(R) \quad \text{for all } x, y \in R \tag{39}$$

and R is commutative by ([11], Corollary 1).
If $d = g = 0$, then putting $y = ys$ in (35) with $0 \neq s \in Z(R) \cap S(R)$ we easily get

$$G(xy) + F(y)F(x) \in Z(R) \quad \text{for all } x, y \in R . \tag{40}$$

Substituting xt for x in the last expression, it follows that

$$G(x)ty + F(y)F(x)t \in Z(R) \quad \text{for all } x, y, t \in R \tag{41}$$

in consequence of which

$$[G(x)ty - G(xy)t, t] = 0 \quad \text{for all } x, y, t \in R \tag{42}$$

that is

$$\Big[G(x)[t, y], t\Big] = 0 \quad \text{for all } x, y, t \in R . \tag{43}$$

Since (43) is the same as (24), we conclude that R is commutative.
Assume that $d = 0$ and $g \neq 0$; replacing y by yh in (35) with
$0 \neq h \in Z(R) \cap H(R)$, we obtain

$$(xy + y^*x^*)g(h) \in Z(R) \quad \text{for all } x, y \in R . \tag{44}$$

The above equation, when combined with Lemma 2.2, shows that R is commutative or $g(z) = 0$ for all $z \in Z(R)$.

In the latter case, replacing y by ys in (35) with $0 \neq s \in Z(R) \cap S(R)$, one can see that

$$G(xy) + F(y)F(x) \in Z(R) \quad \text{for all } x, y \in R . \tag{45}$$

Substituting xt for x in the preceding equation, we find that

$$G(x)ty + xg(ty) + F(y)F(x)t \in Z(R) \quad \text{for all } \ x, y, t \in R \tag{46}$$

and thus

$$[G(x)ty + xg(ty) - G(xy)t, t] = 0 \quad \text{for all } \ x, y, t \in R \tag{47}$$

in consequence of which

$$\Big[G(x)[t, y], t\Big] + [xg(ty) - xg(y)t, t] = 0 \quad \text{for all } \ x, y, t \in R\,. \tag{48}$$

Setting $y = t^2$ in (48), we obviously get

$$[xt^2g(t), t] = 0 \quad \text{for all } \ x, t \in R \tag{49}$$

which yields

$$x[t^2g(t), t] + [x, t]t^2g(t) = 0 \quad \text{for all } \ x, t \in R\,. \tag{50}$$

Writing yx instead of x, we find that

$$[y, t]xt^2g(t) = 0 \quad \text{for all } \ x, y, t \in R \tag{51}$$

and thus

$$[y, t]Rt^2g(t) = 0 \quad \text{for all } \ y, t \in R\,. \tag{52}$$

By primeness we conclude that for each $t \in R$ either $t \in Z(R)$ or $t^2g(t) = 0$.
Let $t \in Z(R)$, substituting yt for y in (45) we obtain

$$xyg(t) \in Z(R) \quad \text{for all } \ x, y \in R \tag{53}$$

in consequence of which, either $xy \in Z(R)$ and hence R is commutative or $g(t) = 0$.
Accordingly, (52) assures that either R is commutative or $t^2g(t) = 0$ for all $t \in R$.

In the second condition, linearization yields that $(x+y)^2g(x+y) = 0$ for all $x, y \in R$.
Let us fix x a nonzero element in $Z(R)$, then we have

$$g(y)x^2 + y^2g(x) + 2yxg(x) + 2yxg(y) = 0 \quad \text{for all } \ y \in R\,. \tag{54}$$

Replacing y by yx in (54), the fact that $x \in Z(R)$ forces

$$g(y) + 2yg(y) = 0 \quad \text{for all } \ y \in R\,. \tag{55}$$

Substituting y^2 for y in (55), we find that

$$g(y^2) = 0 \quad \text{for all } \ y \in R\,. \tag{56}$$

Then by virtu of ([9], Lemma 3) we get $g = 0$, a contradiction.
Now suppose $d \neq 0$ and $g = 0$; replacing y by yh in (35) with
$0 \neq h \in Z(R) \cap H(R)$, we find that

$$(F(x^*)y + y^*F(x))d(h) \in Z(R) \quad \text{for all } \ x, y \in R\,. \tag{57}$$

Writing xh instead of x where h is a nonzero element in $Z(R) \cap H(R)$, we get

$$(xy + y^*x^*)d(h)^2 \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{58}$$

which together with Lemma 2.2, yields that R is commutative or $d(Z(R)) = \{0\}$. In the latter case, replacing y by ys in (35) with $0 \neq s \in Z(R) \cap S(R)$, we get

$$G(xy) + F(y)F(x) \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{59}$$

Substituting xt for x in the above equation, we easily get

$$G(x)ty + F(y)F(x)t + F(y)xd(t) \in Z(R) \quad \text{for all} \quad x, y, t \in R\,. \tag{60}$$

Therefore

$$[G(x)ty - G(xy)t + F(y)xd(t), t] = 0 \quad \text{for all} \quad x, y, t \in R \tag{61}$$

that is

$$\Big[G(x)[t, y], t\Big] + [F(y)xd(t), t] = 0 \quad \text{for all} \quad x, y, t \in R\,. \tag{62}$$

Replacing x with tx and y with t in (62), we find that

$$[F(t)txd(t), t] = 0 \quad \text{for all} \quad x, t \in R\,. \tag{63}$$

Putting $y = t^2$ in (62), we get

$$[F(t^2)xd(t), t] = 0 \quad \text{for all} \quad x, t \in R \tag{64}$$

this equation, when combined with (63), shows that

$$[td(t)xd(t), t] = 0 \quad \text{for all} \quad x, t \in R\,. \tag{65}$$

Substituting xt for x in the above equation, we arrive at

$$[td(t)txd(t), t] = 0 \quad \text{for all} \quad x, t \in R\,. \tag{66}$$

Left multiplying (65) by t and then subtracting from (66), we find that

$$\Big[t[d(t), t]xd(t), t\Big] = 0 \quad \text{for all} \quad x, t \in R\,. \tag{67}$$

Now substituting xt for x, we get

$$\Big[t[d(t), t]xtd(t), t\Big] = 0 \quad \text{for all} \quad x, t \in R\,. \tag{68}$$

Right multiplying (67) by t and then subtracting from (68), we get

$$\Big[t[d(t), t]x[d(t), t], t\Big] = 0 \quad \text{for all} \quad x, t \in R\,. \tag{69}$$

Putting $x = xt$, we have

$$\Big[t[d(t), t]xt[d(t), t], t\Big] = 0 \quad \text{for all } x, t \in R \tag{70}$$

which implies that

$$t[d(t), t]xt[d(t), t]t - t^2[d(t), t]xt[d(t), t] = 0 \quad \text{for all } x, t \in R \,. \tag{71}$$

Substituting $xt[d(t), t]u$ for x in (71), we find that

$$t[d(t), t]xt[d(t), t]ut[d(t), t]t - t^2[d(t), t]xt[d(t), t]ut[d(t), t] = 0 \,. \tag{72}$$

Comparing (71) and (72) shows that

$$t[d(t), t]xt^2[d(t), t]ut[d(t), t] - t[d(t), t]xt[d(t), t]tut[d(t), t] = 0 \tag{73}$$

that is

$$t[d(t), t]x\Big[t[d(t), t], t\Big]ut[d(t), t] = 0 \quad \text{for all } x, t, u \in R \,. \tag{74}$$

Replacing x by tx, we get

$$t[d(t), t]tx\Big[t[d(t), t], t\Big]ut[d(t), t] = 0 \quad \text{for all } x, t, u \in R \,. \tag{75}$$

Left multiplying (74) by t and then subtracting from (75), we find that

$$\Big[t[d(t), t], t\Big]x\Big[t[d(t), t], t\Big]ut[d(t), t] = 0 \quad \text{for all } x, t, u \in R \,. \tag{76}$$

Substituting ut for u in the above equation, we get

$$\Big[t[d(t), t], t\Big]x\Big[t[d(t), t], t\Big]ut^2[d(t), t] = 0 \quad \text{for all } x, t, u \in R \,. \tag{77}$$

Right multiplying (76) by t and then subtracting from (77), we obtain

$$\Big[t[d(t), t], t\Big]x\Big[t[d(t), t], t\Big]u\Big[t[d(t), t], t\Big] = 0 \quad \text{for all } x, t, u \in R \,. \tag{78}$$

Since R is prime, then for each $t \in R$ we have $\Big[t[d(t), t], t\Big] = 0$. Hence ([7], Theorem 2) implies that R is commutative.
(2) $\Longrightarrow$ (5) We are given that

$$G(xx^*) - F(x^*)F(x) \in Z(R) \quad \text{for all } x \in R \tag{79}$$

therefore

$$-G(xx^*) + F(x^*)F(x) \in Z(R) \quad \text{for all } x \in R \,. \tag{80}$$

Define $G' = -G$, the last relation becomes

$$G'(xx^*) + F(x^*)F(x) \in Z(R) \quad \text{for all } x \in R \,. \tag{81}$$

Since (81) is exactly the hypothesis of (1), then R is commutative.
(3) $\Longrightarrow$ (5) Assume that

$$G(xx^*) + F(x)F(x^*) + [x, x^*] \in Z(R) \quad \text{for all} \quad x \in R\,. \tag{82}$$

Linearizing (82), we find that

$$G(xy^*) + G(yx^*) + F(x)F(y^*) + F(y)F(x^*) + [x, y^*] + [y, x^*] \in Z(R)\,. \tag{83}$$

Replacing y by yh, where $h \in Z(R) \cap H(R)\backslash\{0\}$, we obtain

$$(xy^* + yx^*)g(h) + (F(x)y^* + yF(x^*))d(h) \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{84}$$

Substituting xh for x where h is a nonzero element in $Z(R) \cap H(R)$, we arrive at

$$(xy + y^*x^*)d(h)^2 \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{85}$$

Applying Lemma 2.2, we conclude that either R is commutative or $d(Z(R)) = \{0\}$.
Assume that $d(Z(R)) = \{0\}$; then equation (84) reduces to

$$(xy + y^*x^*)g(h) \in Z(R) \quad \text{for all} \quad x, y \in R \tag{86}$$

which together with Lemma 2.2, shows that R is commutative or $g(Z(R)) = \{0\}$.

Suppose that $g(z) = 0$ for all $z \in Z(R)$; replacing y by ys in (83) where s is a nonzero element in $Z(R) \cap S(R)$, we get

$$G(xy) + F(x)F(y) + [x, y] \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{87}$$

Using ([11], Theorem 8), we conclude that R is commutative or $d = g = 0$.
Assume that $d = g = 0$; substituting yt for y in (87) we obtain

$$G(xy)t + F(x)F(y)t + y[x, t] + [x, y]t \in Z(R) \quad \text{for all} \quad x, y, t \in R\,. \tag{88}$$

Commuting with t and invoking (87), it is easy to see that

$$\Big[y[x, t], t\Big] = 0 \quad \text{for all} \quad x, y, t \in R \tag{89}$$

and thus

$$y\Big[[x, t], t\Big] + [y, t][x, t] = 0 \quad \text{for all} \quad x, y, t \in R\,. \tag{90}$$

Putting $y = uy$ in the above equation, one can verify that

$$[u, t]y[x, t] = 0 \quad \text{for all} \quad x, y, t, u \in R\,. \tag{91}$$

In light of primeness, equation (91) shows that R is commutative.
Using a similar proof, with slight modifications, it is easy to show that

$G(xx^*) + F(x)F(x^*) - [x, x^*] \in Z(R)$ for all $x \in R$ implies that R is commutative.
(4) $\Longrightarrow$ (5) Suppose that

$$G(xx^*) - F(x)F(x^*) \pm [x, x^*] \in Z(R) \quad \text{for all} \quad x \in R\,. \tag{92}$$

Hence

$$-G(xx^*) + F(x)F(x^*) \pm [x, x^*] \in Z(R) \quad \text{for all} \quad x \in R\,. \tag{93}$$

Let $G' = -G$, the last relation yields

$$G'(xx^*) + F(x)F(x^*) \pm [x, x^*] \in Z(R) \quad \text{for all} \quad x \in R\,. \tag{94}$$

Since (94) is exactly the hypothesis of (3), then R is commutative. □

Corollary 2.2. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If (F, d) and (G, g) are nonzero generalized derivations, then the following assertions are equivalent:*
(1) $G(xy) \pm F(y)F(x) \in Z(R)$ for all $x, y \in R$;
(2) $G(xy) \pm F(x)F(y) \pm [x, y] \in Z(R)$ for all $x, y \in R$
(3) R is commutative.

Theorem 2.3. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If (F, d) and (G, g) are generalized derivations where $d \neq 0$ or $g \neq 0$, then the following assertions are equivalent:*
(1) $G(xx^*) + F(x)F(x^*) \in Z(R)$ for all $x \in R$
(2) $G(xx^*) - F(x)F(x^*) \in Z(R)$ for all $x \in R$
(3) $G(xx^*) + F(x^*)F(x) \pm [x, x^*] \in Z(R)$ for all $x \in R$
(4) $G(xx^*) - F(x^*)F(x) \pm [x, x^*] \in Z(R)$ for all $x \in R$
(5) R is commutative.

Proof. We need only prove (1) $\Longrightarrow$ (5), (2) $\Longrightarrow$ (5), (3) $\Longrightarrow$ (5), and (4) $\Longrightarrow$ (5).
(1) $\Longrightarrow$ (5) Assume that

$$G(xx^*) + F(x)F(x^*) \in Z(R) \quad \text{for all} \quad x \in R\,. \tag{95}$$

Linearizing (95), we find that

$$G(xy^*) + G(yx^*) + F(x)F(y^*) + F(y)F(x^*) \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{96}$$

Taking $y = yh$ in (96), where $h \in Z(R) \cap H(R)\backslash\{0\}$, we get

$$(xy^* + yx^*)g(h) + (F(x)y^* + yF(x^*))d(h) \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{97}$$

Substituting xh for x, where $h \in Z(R) \cap H(R)\backslash\{0\}$, we obtain

$$(xy + y^*x^*)d(h)^2 \in Z(R) \quad \text{for all} \quad x, y \in R\,. \tag{98}$$

Invoking Lemma 2.2, R is commutative or $d(Z(R)) = \{0\}$.
In the last case, equation (97) reduces to

$$(xy + y^*x^*)g(h) \in Z(R) \quad \text{for all } \ x, y \in R\,. \tag{99}$$

Applying Lemma 2.2 again, we conclude that R is commutative or $g(Z(R)) = \{0\}$.
Assume that $g(z) = 0$ for all $z \in Z(R)$; replacing y by ys in (96) where s is a nonzero element in $Z(R) \cap S(R)$, we obviously get

$$G(xy) + F(x)F(y) \in Z(R) \quad \text{for all } \ x, y \in R \tag{100}$$

hence R is commutative by ([11], Corollary 5).
(2) $\Longrightarrow$ (5) Suppose

$$G(xx^*) - F(x)F(x^*) \in Z(R) \quad \text{for all } \ x \in R \tag{101}$$

therefore

$$-G(xx^*) + F(x)F(x^*) \in Z(R) \quad \text{for all } \ x \in R\,. \tag{102}$$

Let G' be the generalized derivation defined by $G' = -G$, the last relation becomes

$$G'(xx^*) + F(x)F(x^*) \in Z(R) \quad \text{for all } \ x \in R\,. \tag{103}$$

Since (103) is exactly the hypothesis of (1), then R is commutative.
(3) $\Longrightarrow$ (5) Suppose that

$$G(xx^*) + F(x^*)F(x) + [x, x^*] \in Z(R) \quad \text{for all } \ x \in R\,. \tag{104}$$

Linearization of (104) yields

$$G(xy^*) + G(yx^*) + F(x^*)F(y) + F(y^*)F(x) + [x, y^*] + [y, x^*] \in Z(R)\,. \tag{105}$$

Replacing y by yh, where $h \in Z(R) \cap H(R)\backslash\{0\}$, we get

$$(xy^* + yx^*)g(h) + (F(x^*)y + y^*F(x))d(h) \in Z(R) \quad \text{for all } \ x, y \in R\,. \tag{106}$$

If we set $x = xh$ where $h \in Z(R) \cap H(R)\backslash\{0\}$, then we obtain

$$(xy + y^*x^*)d(h)^2 \in Z(R) \quad \text{for all } \ x, y \in R \tag{107}$$

which when combined with Lemma 2.2, shows that R is commutative or $d(Z(R)) = \{0\}$.
Assume that $d(Z(R)) = \{0\}$; putting $y = ys$ in (106) where s is a nonzero element in $Z(R) \cap S(R)$, we arrive at

$$(xy + y^*x^*)g(h) \in Z(R) \quad \text{for all } \ x, y \in R\,. \tag{108}$$

Once again using Lemma 2.2, we conclude that R is commutative or $g(Z(R)) = \{0\}$.
Suppose that $g(z) = 0$ for all $z \in Z(R)$; replacing y by ys in (105) with
$0 \neq s \in Z(R) \cap S(R)$ thereby obtaining

$$G(xy) + F(y)F(x) + [x, y] \in Z(R) \quad \text{for all } \ x, y \in R\,. \tag{109}$$

Hence ([11], Theorem 7) proves that R is commutative.
Using a similar proof, with slight modifications, one can easily prove that
$G(xx^*) + F(x^*)F(x) - [x, x^*] \in Z(R)$ for all $x \in R$ implies that R is commutative.
(4) $\Longrightarrow$ (5) We are given that

$$G(xx^*) - F(x^*)F(x) \pm [x, x^*] \in Z(R) \quad \text{for all} \quad x \in R \tag{110}$$

therefore

$$-G(xx^*) + F(x^*)F(x) \pm [x, x^*] \in Z(R) \quad \text{for all} \quad x \in R\,. \tag{111}$$

Set $G' = -G$, from the last relation it follows that

$$G'(xx^*) + F(x^*)F(x) \pm [x, x^*] \in Z(R) \quad \text{for all} \quad x \in R\,. \tag{112}$$

Since (112) is exactly the hypothesis of (3), then R is commutative. □

Corollary 2.3. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If (F, d) and (G, g) are generalized derivations where $d \neq 0$ or $g \neq 0$, then the following assertions are equivalent:*
(1) $G(xy) \pm F(x)F(y) \in Z(R)$ for all $x, y \in R$
(2) $G(xy) \pm F(y)F(x) \pm [x, y] \in Z(R)$ for all $x, y \in R$
(3) R is commutative.

The following example proves that the condition "* is of the second kind" is necessary in our Theorems.

Example 1. *Let us consider* $R = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}\right\}$ *and* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. *It is straightforward to check that $(R, *)$ is a prime ring with involution of the first kind. Moreover, for all $x \in R$ we have*

$$xx^* = x^*x \in Z(R)$$

Let us define $F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$, *then F is a generalized derivation associated with the nonzero derivation d. Furthermore, if we take $G = F$, then the conditions of our theorems are satisfies, but R is a noncommutative prime ring.*

The following example shows that our results cannot be extended to semi-prime rings.

Example 2. *Let us consider $(R, *)$, F and G as in the preceding example. Let $\mathbb{C}$ be the field of complex numbers with the conjugation involution. If we set $\mathcal{K} = R \times \mathbb{C}$, then it is obvious to verify that $(\mathcal{K}, \tau)$ is a semi-prime ring with involution of the second kind where*

$$\tau(r, z) = (r^*, \overline{z}) \quad \textit{for all} \quad (r, z) \in R \times \mathbb{C}\,.$$

It is straightforward to check that the map

$$\begin{aligned} \mathcal{F}\colon \mathcal{K} &\longrightarrow \mathcal{K} \\ (r, z) &\longmapsto (F(r), 0) \end{aligned}$$

is a generalized derivation associated with the derivation

$$\begin{aligned} D\colon \mathcal{K} &\longrightarrow \mathcal{K} \\ (r,z) &\longmapsto (d(r),0) \end{aligned}$$

On the other hand, if we put $\mathcal{G} = \mathcal{F}$, *then it is easy to verify that* $\mathcal{G}$ *and* $\mathcal{F}$ *satisfy the conditions of our Theorems but* $\mathcal{K}$ *is a noncommutative ring.*

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